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FRANK MORLEY, EDITOR

A. COHEN, ASSOCIATE EDITOR

WITH THE COOPERATION OF

CHARLOTTE A. SCOTT, A. B. COBLE

AND OTHER MATHEMATICIANS

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L'expression la plus générale de la "distance" sur une droite.

PAR MAURICE FRÉCHET.
(*Université de Strasbourg*)

Position du problème. Le problème que nous nous proposons de résoudre est le suivant:

Etant donnée une droite indéfinie, attacher à tout couple de points de cette droite, d'abscisses x, x' un nombre (x, x') qu'on pourra appeler distance généralisée de x et de x' satisfaisant aux conditions suivantes.

1° On a $(x, x') = (x', x) \geq 0$.

2° Le nombre (x, x') est positif si x, x' sont distincts et nul s'ils coïncident.

3° Quels que soient les points x, x', x'', on a

$$(x', x'') \leq (x, x') + (x, x'').$$

4° La condition nécessaire et suffisante pour qu'une suite de points $x_1, x_2, \cdots, x_n, \cdots$, convergent vers un point x est que la suite de nombres $(x, x_1), (x, x_2), \cdots, (x, x_n), \cdots$, converge vers zéro.

Un solution évidente du problème consiste à prendre

$$(1) \qquad (x, x') = |x - x'| \times \text{constante}.$$

Mais nous nous proposons de *trouver l'expression la plus générale de la distance généralisée.*

Solutions particulières. Il est manifeste que la solution que nous venons d'obtenir n'est, en effet, pas la seule. Si par exemple on fait correspondre à la droite x, x' biunivoquement et bicontinuement une courbe de Jordan sans points multiples (extrémités exclues, s'il en existe), la distance géométrique entre les deux points X', X'' de cette courbe qui correspondent aux deux points x', x'' de Δ, vérifiera les propriétés 1°, 2°, 3°, 4°. Par consé-

quent, on pourra prendre pour distance généralisée (x', x'') la distance géométrique de X' et X''.

Cette solution est essentiellement distincte de la solution (1), si la courbe de Jordan n'est pas une ligne droite; en effet dans ce cas, on aura pour trois valeurs x', x'', x''' de x correspondant à trois points non alignés X', X'', X'''.

$$| (x', x'') - (x', x''') | < (x'', x''') < (x', x'') + (x', x''')$$

alors que dans le cas de la solution (1) l'un des signes d'inégalité serait remplacé par un signe d'égalité.

Cette remarque montre en outre que contrairement à ce qu'il serait peut être naturel de supposer la solution générale ne s'obtient pas en transformant la droite Δ en elle même par une transformation biunivoque et bicontinue et prenant pour (x', x'') la distance géométrique des points correspondants à x' et x''.

Propriétés de la distance généralisée. Il ne sera pas inutile d'établir deux propriétés de la distance généralisée qui découlent naturellement de sa définition.

Tout d'abord la distance généralisée (x', x'') est bornée dans tout intervalle fini I. Car dans le cas contraire, on pourrait trouver dans I, x'_p, x''_p tels que

$$(x'_p, x''_p) > p.$$

Et on pourrait extraire de la suite des couples x'_p, x''_p une suite convergeant vers le couple t'_0, t''_0. On aurait donc d'après 3°, si après un changement de notation convenable, on suppose que ceci a lieu pour la suite donnée:

$$p < (x'_p\, x''_p) \leq (x'_p, t'_0) + (t'_0, t''_0) + (t''_0, x''_p);$$

le second membre qui converge d'après 4° vers (t'_0, t''_0) ne pourrait donc rester supérieur à p.

En outre la distance généralisée tend uniformément vers zéro dans I quand $| x' - x'' |$ tend vers zéro. Autrement il existerait un nombre $\epsilon > 0$ tel que quel que soit p, il existe un couple x'_p, x''_p tel que

$$| x'_p - x''_p | < 1/p, \quad (x'_p, x''_p) > \epsilon.$$

Comme précédemment on peut supposer que le couple x'_p, x''_p a une limite

(t'_0, t''_0) déterminée; mais cette fois on aura nécessairement $t'_0 = t''_0$. Alors:

$$0 < \epsilon < (x'_p, x''_p) \leq (x'_p, t'_0) + (t''_0, x''_p)$$

et le second membre tendant vers zéro, ne pourrait rester supérieur à ϵ.

La distance généralisée (x', x'') est une fonction des deux variables x', x''. C'est une fonction uniformément continue de x', x'' dans tout intervalle I. Autrement dit, $\epsilon > 0$ etant donné, on peut trouver un nombre $\eta > 0$, tel que les inégalités $| x' - y' | < \eta$, $| x'' - y'' | < \eta$ vérifiées dans I entraînent

$$| (x', x'') - (y', y'') | < \epsilon.$$

En effet, on déduit de 3°

$$| (x', x'') - (y', y'') | \leq (x', y') + (x'', y'')$$

et d'après la propriété précédemment établie, on peut trouver η tel que les inégalités

$$| x' - y' | < \eta, \quad | x'' - y'' | < \eta \qquad \text{vérifiées dans I}$$

entraînent

$$(x', y') < \eta/2, \quad (x'', y'') < \eta/2.$$

Recherche de la solution générale du problème. Nous avons vu qu'on pouvait former des solutions du problème au moyen de courbes de Jordan mises en correspondance avec Δ. Mais le raisonnement que nous avons fait garde sa valeur quel que soit le nombre de dimensions de l'espace dans lequel est tracé cette courbe.

Cette remarque nous conduit à chercher si l'on pourrait obtenir la solution du problème en utilisant une courbe de Jordan tracée dans un espace à une infinité de coordonnées, c'est à dire où chaque point X est défini par une suite infinie de nombres réels $X_1, X_2, \cdots, X_n, \cdots$, et où on a adopté un sens déterminé pour la notion de suite convergente de points $X^{(1)}, X^{(2)}, \cdots, X^{(p)}, \cdots$, de cet espace.

On sait que suivant les applications qu'on a en vue plusieurs définitions non équivalentes des suites convergentes peuvent être envisagées. Notre choix se portera sur la définition qui fournit l'espace D_ω, pour profiter d'une propriété de l'espace D_ω qui nous sera utile,* et que nous démontrerons ici.

L'espace D_ω est un espace à une infinité de coordonnées où une suite

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* *Rendiconti del Circole Matematico di Palermo*, t. 30, 1910, p. 12.

de points $X^{(1)}, X^{(2)}, \cdots, X^{(n)}, \cdots$, est dite converger vers un point X lorsque les coordonnées de $X^{(n)}$ tendent uniformément vers les coordonnées de même rang de X.

On peut définir dans cet espace une " distance " satisfaisant aux conditions 1°, 2°, 3°, 4°, en appelant distance de deux points X, X' de D_ω et en désignant par (X, X') la borne supérieure des valeurs absolues $|X_n - X'_n|$ des différences des coordonnées de même rang de X et de X'.

Pour éviter d'introduire des distances infinies actuelles, il sera commode de limiter D_ω aux points dont les coordonnées sont bornées (leurs bornes pouvant varier avec chaque point).

Solution générale du problème. L'expression (x, x') la plus générale de la " distance généralisée " de deux points x, x' d'une droite Δ s'obtient de la façon suivante: ou considére une courbe de Jordan sans points multiples de l'espace D_ω. Par définition, il existe une correspondance biunivoque et bicontinue entre C et la droite Δ; si X, X' sont les points de C qui correspondent aux points x, x' de Δ, on prendra pour (x, x'), la distance (X, X'), telle qu'elle a été definie plus haut.

1° Supposons donnée l'expression (x, x') de la distance généralisée sur la droite Δ. Nous allons définir la courbe C. Pour cela, suivant le procédé cité ci-dessus,* nous appellerons $t_1, t_2, \cdots, t_n, \cdots$, les abscisses d'un ensemble dénombrable partout dense de points de la droite Δ. (Par exemple, ce seront les abscisses rationnelles), et nous définirons C par les expressions suivantes de coordonnées $x_1, x_2, \cdots$, d'un point variable X de C:

$$(1) \quad X_1 = (x, t_1), \quad X_2 = (x, t_2) - (t_2, t_1), \cdots, \quad X_n = (x, t_n) - (t_n, t_1), \cdots$$

A chaque point de coordonnée x de Δ correspond un point X de C, car non seulement les coordonnées de X sont numériquement déterminées, mais pour une valeur déterminée de x, ces coordonnées sont bornées, puisque d'après 3°, on a quelque soit n

$$|X_n| \leq (x, t_1)$$

La distance (X, X') de deux points X, X' de C est par définition la borne supérieure de

$$|X_n - X'_n| = |(x, t_n) - (x', t_n)|.$$

* *Rendiconti del Circole Matematico di Palermo*, t. 30, 1910, p. 12.

Or d'après 3°, le second membre est au plus égal à (x, x') qui est indépendant de n, donc

$$(X,\ X') \leq (x, x'). \tag{2}$$

Puisque l'ensemble $t_1, t_2, \cdots$, est dense sur Δ, on peut extraire de cet ensemble une suite $t_{q_1}, t_{q_2}, \cdots$, qui converge vers x':

D'après 3°

$$|\ (x, x')-(x', t_{q_i})\ | \leq (x, t_{q_i}) \leq (x, x')+(x', t_{q_i})$$

Lorsqu'on tient compte de 4° ces deux inégalités donnent à la limite:

$$(x, x') = \lim_{i\to\infty} (x, t_{q_i}).$$

L'égalité

$$|\ X_{q_i} - X'_{q_i}\ | = |\ (x, t_{q_i}) - \ (x', t_{q_i})\ |$$

donne donc

$$(x, x') = \lim_{i\to\infty} |\ X_{q_i} - X'_{q_i}\ |$$

et comme

$$(X, X') \geq |\ X_q - X'_q\ |$$

quelque soit q on aura

$$(X, X') \geq (x, x'), \tag{3}$$

De (2) et (3), on tire enfin:

$$(X, X') = (x, x'). \tag{4}$$

On en déduit d'abord qu'à tout point X de C ne correspond qu'un point x de Δ; car s'il en correspondait un autre $x' \neq x$, on pourrait supposer dans l'égalité (4) que X' est identique à X et d'après 1°, que le premier membre est nul et le second positif. Ainsi C n'a pas de points multiples, la correspondance entre C et Δ est biunivoque. Elle est aussi bicontinue. En effet si X' tend vers X, (X, X') tend vers zéro, donc aussi (x, x') et par suite, d'après 4° x' tend vers x. Réciproquement, si x' tend vers x, (x, x') d'après 4° tend vers zéro; donc aussi (X, X') et par suite X' tend vers X.

D'après cela la courbe C, représentée par les équations (1) est représentée paramétriquement sous la forme

$$x_1 = X_1(x), \quad x_2 = X_2(x), \cdots,$$

où $X_1(x)$, $X_2(x)$, $\cdots$, sont des fonctions individuellement continues de x. En effet quel que soit n

$$| X_n(x) - X_n(x') | \leq (X, X') = (x, x')$$

et si x' tend vers x, (x', x) tend vers zéro et $X_n(x')$ tend vers $X_n(x)$. On doit remarquer de plus que les fonctions $X_1(x)$, $X_2(x)$, $\cdots$, sont bornées dans leur ensemble et également continues dans tout intervalle partiel I de la droite Δ. En effet

$$| X_n(x) | = | (x, t_n) - (t_n, t_1) | \leq (x, t_1)$$

et si x reste dans un intervalle fini, t_1 restant fixe, on a vu que (x, t_1) est borné.

Les fonctions $X_n(x)$ sont aussi également continues dans I, car

$$| X_n(x) - X_n(x') | \leq (X, X') = (x, x')$$

et nous avons vu que si $\epsilon > 0$ est donné, il existe un nombre $\eta > 0$, tel que dans I

$$\text{pour } | x - x' | < \eta \text{ on ait } (x, x') < \epsilon \text{ et par suite.}$$
$$| X_n(x) - X_n(x') | < \epsilon \quad \text{quel que soit } n.$$

Enfin remarquons que les extrémités de C s'il y en a sont exclues de la correspondance. Par extrémité de C nous entendons tout point de l'espace D_ω qui serait point limite de points de C correspondant à des points x_1, x_2, $\cdots$, de Δ qui tendent vers $+\infty$ ou vers $-\infty$. Il s'agit de montrer que s'il existe un tel point limite, il ne coïncide pas en position avec un point $X(x)$ de C correspondant à une abscisse finie de x. En effet, on aurait $(x, x_n) = (X(x), X(x_n))$ et le second membre tendant vers zéro (x, x_n) tendrait vers zéro et par suite x_n tendrait vers un nombre fini x.

Ceci montre que si la définition de la distance (x, x') sur la droite Δ est compatible avec une généralisation du critère de convergence de Cauchy, la courbe C est illimitée dans les deux sens. Sans quoi, il y aurait une suite de points de Δ $x_1, x_2, \cdots, x_n, \cdots$, tendant vers $+\infty$ ou vers $-\infty$ telle que la suite des $X(x_n)$ vérifie la condition de convergence de Cauchy. Donc aussi la suite des x_n puisque $(X(x_n), X(x_{n+p})) = (x_n, x_{n+p})$. Et alors la suite des x_n devrait être convergente.

2° Réciproquement considérons une courbe C quelconque de l'espace D_ω qu'on peut mettre en correspondance biunivoque et bicontinue (extrémités de C, s'il y en a exclues) avec la droite Δ. Si x est l'abscisse du point de Δ qui correspond au point X de C, les coordonnées de X seront des fonctions uniformes de x.

$$X_1 = X_1(x), \quad X_2 = X_2(x), \cdots,$$

et l'ensemble de ces fonctions ne peut prendre deux ensembles identiques de valeurs pour deux valeurs différentes de x.

Ceci étant, posons

$$(x, x') = (X(x), X(x'))$$

et montrons qu'on peut considérer (x, x') comme une solution du problème, c'est à dire qu'elle satisfait aux conditions 1°, 2°, 3°, 4°.

Or cela résulte: d'une part de ce que des conditions semblables sont vérifiées pour deux points quelconques X, X' de la courbe sans points multiples, V (et même pour deux points quelconques de l'espace D_ω comme je l'ai montré précédemment); d'autre part de ce que la correspondance entre C et Δ est biunivoque et bicontinue.

Dans le cas où la courbe C est illimitée dans les deux sens, la distance (x, x') ainsi définie est compatible avec une généralisation du critère de convergence de Cauchy.

Forme analytique de la solution. On peut aussi donner à la solution générale que nous venons d'obtenir une forme purement analytique, débarassée d'une terminologie géométrique (qui a pourtant l'avantage de présenter l'énoncé sous une forme plus ramassée et plus intuitive).

Il suffit pour cela de chercher à quelles conditions doivent satisfaire les fonctions $X_1(x)$, $X_2(x)$, $\cdots$, pour qu'elles puissent représenter les coordonnées d'une courbe de Jordan (extrémités, s'il y en a, exclues) sans points multiples, dans l'espace D_ω.

Il faut d'abord que l'ensemble des fonctions $X_1(x)$, $X_2(x)$ $\cdots$, supposées uniformes ne puisse prendre le même ensemble de valeurs pour deux valeurs de x distinctes.

Pour que le point $X_1(x)$, $X_2(x)$, $\cdots$, appartienne à D_ω, il faut ensuite que les fonctions de cette suite soient bornées dans leur ensemble pour chaque valeur de x considérée isolément.

On peut même voir qu'elles sont bornées dans leur ensemble dans tout intervalle partiel I. Car dans le cas contraire il existerait quel que soit p, une valeur x_p dans I et une fonction X_{n_p}, telle que $| X_{n_p}(x_p) | > p$. On peut comme précédemment supposer que la suite des x_p converge vers un point t_0 de I. Or on a

$$p < | X_{n_p}(x_p) | \leq | X_{n_p}(x_p) - X_{n_p}(t_0) | + | X_{n_p}(t_0) | \leq (X(x_p), X(t_0)) + | X_{n_p}(t_0) | ;$$

et le dernier membre qui devrait ainsi croître au dela de toute limite est la somme de deux quantités l'une tendant vers zéro l'autre bornée quand p varie.

Enfin les fonctions $X_1(x)$, $X_2(x)$, $\cdots$, sont aussi également continues dans tout intervalle I; il suffit de montrer que quel que soit ϵ, il existe un nombre η tel que dans I l'inégalité

$$| x - x' | < \eta \text{ entraîne } \quad (X(x), X(x')) < \epsilon$$

puisque $| X_n(x) - X_n(x') | \leq (X(x), X(x'))$ quelque soit n. Or dans le cas contraire il existerait quelque soit p un couple x_p, x'_p tel que

$$| x_p - x'_p | < 1/p \qquad (X(x_p), X(x'_p)) \geq p.$$

On peut supposer comme précédemment que la suite des x'_p converge vers un point t'_0 de I. Mais alors il en est de même des x_p et comme on a

$$p \leq (X(x_p), X(x'_p)) \leq (X(x_p), X(t'_0)) + (X(x'_p), X(t'_0))$$

le second membre qui tend vers zéro devrait croître infiniment.

Réciproquement les trois propriétés des X_n que nous venons de reconnaître comme nécessaires sont aussi suffisantes.

Le fait que les $X_n(x)$ sont bornées dans leur ensemble pour chaque valeur de x, montre que $X_1(x)$, $X_2(x)$, $\cdots$, sont les coordonnées d'un point $X(x)$ bien déterminé de D_ω.

La première propriété assure ensuite la biunivocité de la correspondance entre x de Δ et $X(x)$ de C.

Quand à la bicontinuité, nous remarquons que les fonctions $X_n(x)$ étant également continues dans tout intervalle I, à tout nombre $\epsilon > 0$ correspond un nombre $\eta > 0$, tel que dans I l'inégalité $| x - x' | < \epsilon$ entraîne $| X_n(x) - X_n(x') | < \eta$ et aussi en prenant la borne supérieure du premier membre quand n varie

$$(X(x), X(x')) < \eta$$

Par conséquent quand on prend pour I un intervalle fixe de centre x et qu'on fait tendre x' vers x, $X(x')$ tendra vers $X(x)$.

En ce qui concerne la réciproque, il faut montrer que si $X(x_p)$ tend vers $X(x)$, x_p tend vers x. En effet supposons d'abord que tous les x_p restent dans un intervalle fini I. Alors si x' est la plus grande (ou la plus petite) des limites de x_p pour p infini, il y a un suite $x_{p_1}, x_{p_2}, \cdots$, extraite des x_p qui tend vers x'; par suite comme nous l'avons vu $X(x_{p_q})$ tend vers $X(x')$ et aussi par hypothèse vers $X(x)$; donc $x = x'$. Comme ceci s'applique à la fois à la plus grande et à la plus petite des limites, on voit finalement que la suite elle même des x_p est convergente et tend vers x.

Reste le cas où l'on pourrait extraire des x_p une suite $x_{p_1}, x_{p_2}, \cdots$, qui converge vers $+\infty$ ou vers $-\infty$, par exemple vers $+\infty$. Alors le point $X(x)$ qui serait le point limite dans D_ω des point $X(x_{p_1}), X(x_{p_2}), \cdots$, d'une part appartiendrait à la courbe C, d'autre part étant une extrémité de C ne lui appartiendrait pas.

Par conséquent, la solution générale du problème posé peut s'exprimer analytiquement de la façon suivante.

Pour définir une distance généralisée sur une droite indéfinie Δ, on peut toujours procéder ainsi: on appellera distance généralisée de deux points x, x' la borne supérieure quand n varie de

$$| X_n(x) - X_n(x') |$$

en désignant par $X_1(x), X_2(x), \cdots, X_n(x), \cdots$, une suite de fonctions uniformes qui sont sur tout intervalle fini bornées dans leur ensemble et également continues et telles de plus 1° qu'elles ne prennent pas le même ensemble de valeurs pour deux valeurs de x distinctes 2° que s'il existe une suite de valeurs de x, $x_1, x_2, \cdots$, tendant vers $+\infty$ ou vers $-\infty$ sur lesquelles les fonctions $X_1(x), X_2(x), \cdots$, convergent uniformément, l'ensemble des valeurs limites ne puisse être pris en un point de Δ.

Remarques I. Une méthode tout à fait analogue réussirait sans doute si Δ au lieu d'être une droite était un plan ou l'espace euclidien à trois dimensions, etc. . . .

II. Il serait intéressant de rechercher si dans ce qui précéde on pourrait remplacer l'espace D_ω par un espace plus simple. Il est facile de voir qu'on ne peut le remplacer par un espace à un nombre fini n de dimensions. Pour

un tel espace en effet, il existe une relation entre les distances de $n+2$ points qui n'a pas lieu nécessairement pour un courbe quelconque tracée dans un espace à une dimension de plus. Mais peut-être pourrait-on remplacer D_ω qui n'est pas séparable par un espace (à une infinité de dimensions) qui serait séparable?

21 Août 1924.

Second Paper on Tensor Analysis.

By G. Y. Rainich.

In this paper which forms the continuation of a paper entitled "Two-Dimensional Tensor Analysis Without Coördinates" (This Journal, Vol. XLVI, No. 2, April, 1924, pp. 71-94) we shall follow in the main the same course as in the paper mentioned above but instead of considering a 2-dimensional surface immersed in a 3-dimensional Euclidean space we shall consider an n-dimensional curved space Σ immersed in an N-dimensional Euclidean space. It will be convenient for us to continue throughout all these papers the numbering of the sections which we started in the first paper and we shall refer to the sections and formulas of the preceding paper without mentioning its title.

As in the case of the first paper, most of the results are known,* and the main purpose is to *simplify* the theory and to eliminate all that is unnecessary.

§ 14. *Generalization of preliminary notions.*

We shall use as an abbreviation for N-dimensional Euclidean space simply N-E.S., for n-dimensional space (which in general will be curved), we shall write n-S.; these will be only *abbreviations* and *not mathematical symbols;* 2-S. will be another *word* for surface, 3-E.S. another *word* for space (ordinary), 1-S. another word for curve etc. When we wish to refer to a particular space we use letters or letters with subscripts.

We shall have here to deal with points and vectors in a N-E.S., but the operations and notations remain the same as in a 3-E.S.: we use capital Roman letters for points, small Roman letters for vectors, which we consider as differences between points. The vectors are freely moveable in the space (unless the opposite is explicitly stated); there are therefore no new difficulties connected with addition and multiplication of vectors, because we always can bring their initial points together, and then both vectors are in the same plane.

We shall often have to use projections (i. e. orthogonal projections) of points and vectors on an n-E.S., and we shall use the following.

* Compare for bibliography the article by J. A. Schouten and D. J. Struik in the 46 Vol. of *Palermo Rendiconti*, p. 162.

Remark. If an r-E.S. lies entirely in an n-E.S. (of course, $r < n$) then instead of projecting a point or a vector on this r-E.S. we may first project it on the containing n-E.S. and after this project the projection on the r-E.S. considered.

The product of two vectors x and y may be considered as the product of the length of x by the length of the projection of y on the direction of x; it follows from this and the foregoing remark that

14.1 $$x.y = x.\bar{y}$$

where $\bar{y}$ is the projection of y on any E.S. which contains x.

The definition of a linear function as a function $\phi(x)$, which satisfies the equation $\phi(x+y) = \phi(x) + \phi(y)$, holds here; a tensor is again a multilinear function and a *flat* tensorfield is an immediate generalization of a plane tensorfield (§ 4): if a function $\phi(A, x, y, \cdots)$ with one point argument A, which is a point of an n-E.S., and several vector arguments $x, y, \cdots$ which are vectors of the same n-E.S. is linear with respect to the vector arguments, we call it a flat tensorfield.

In § 1 we considered an axis and a curve in a plane, or as we shall say now, a 1-E.S. and a 1-S. in a 2-E.S.; in § 2 we considered a plane Π and a surface, say Σ, in the ordinary space—or as we shall say now: a 2-E.S. and a 2-S. in a 3-E.S. We shall consider now an n-E.S. Π and an n-S. Σ in an N-E.S. (an n-dimensional flat, or Euclidean, space Π and an n-dimensional cuved space Σ in an N-dimensional Euclidean space). We shall consider together with the points of Π the points of Σ whose (orthogonal) projections they are; in the cases considered before, all the vectors along which the points are projected (the vectors which have the projections for their initial points and the projected points for their tips) are parallel to each other and their direction was therefore always known, it was *the* direction perpendicular to Π; it was therefore enough to give the length (and the sense) of the *projecting vector.* Things are different when $N - n > 1$—a good illustration is the case of a curve in ordinary space where $N = 3$, $n = 1$; let us take for Σ a helix and for Π its axis; to each point A of Π there corresponds a point of Σ whose projection is A, but the projecting vectors have not all the same directions: as A moves on the axis, the projecting vector rotates; i. e. changes its direction. It will not suffice to give the length of the projecting vector $\phi(A)$; we shall have to give its direction as well; i. e. we shall have to consider the whole vector. To give Σ we shall have to consider at each point A of Π the vector—we shall designate it $f(A)$—which leads from the point A to the point of Σ whose projection is A. If in the case of a surface we

had designated the unit perpendicular vector to Π by p, what we now designate by $f(A)$ would be equal to $p.\phi(A)$. In the general case $f(A)$ answers every purpose which we required of $\phi(A)$ in the case of the surface. The function $f(A)$ can be said to give us a representation of Σ in the sense that the point of Σ whose projection on Π is A is here given by

$$A + f(A).$$

The n-E.S., tangent to Σ at that point (if it exists), will be given by

$$A + h + f(A) + f'(A, h),$$

where $f'(A, h)$ is the differential of $f(A)$, and may be found (if it exists) by aid of the formula

$$f'(A, h) = \lim_{\lambda \to 0} \frac{f(A + \lambda h) - f(A)}{\lambda} \quad ,$$

which corresponds to 2.3; the only difference being that here $f'(A, h)$, $f(A + \lambda h)$, and $f(A)$ are vectors perpendicular to Π, whereas in 2.3 we considered only the length of these vectors, they having a fixed direction. As in that case, the differential $f'(A, h)$ is a linear function of h; but it is a vector function now and cannot be considered as a product.

§ 15. *The indicator.*

We suppose that the n-S. Σ, which we are considering has a definite tangent n-E.S. at each point, and we shall consider the vectors of this n-E.S. at the point A as vectors of the bundle A, which belongs to Σ; they will be denoted by x_A, y_A etc.

We agree, as before, to denote by B_A the projection of the point B of Σ on the n-E.S. which is tangent to Σ at the point A and put

$$B - B_A = f(B_A);$$

$f(B_A)$ is then the projecting vector considered in the preceding section, and

15.1 $$B_A + f(B_A)$$

is the representation of Σ, with the tangent n-E.S. at A playing the rôle of Π; and

15.2 $$B_A + h_A + f(B_A) + f'(B_A, h_A)$$

is the representation of the n-E.S. tangent at B. If we make $B = A$ we have, as in 6.2,

$$f(A) + f'(A, h_A) = 0$$

whence

15.3 $$f(A) = 0, \qquad f'(A, h_A) = 0,$$

We assume now that the second differential of $f(B_A)$ exists at the point A and denote it by $s(h_A, k_A)$:

15.4 $$s(h_A, k_A) = f''(A, h_A, k_A).$$

The so defined $s(h_A, k_A)$ is a bilinear function whose arguments are vectors of the bundle A and whose values are vectors perpendicular to the bundle A, or normal vectors at A. This corresponds not to what we called the indicator, viz., $\sigma(h_A, k_A)$ in the case $n = 2$, $N = 3$ but to $a.\sigma(h_A, k_A)$ with a the normal unit vector; of course, before it was enough to know $\sigma(h_A, k_A)$, i. e. the length of $s(h_A, k_A)$, because the direction was *the* direction of the normal. Here we have to consider $s(h_A, k_A)$, and we shall call *this* the indicator.* It is not to be confused with $s(h_A)$—one of the factors, into which $\sigma(h_A, k_A)$ could be split up—there is no such thing unless $N - n = 1$.

Now, all we did for the point A, we can do for every point of Σ, and we have an indicator at every point.

We have thus a function which has a point argument A and two vector arguments h_A and k_A, upon each of which it depends linearly. Still it does not correspond to what we called a tensorfield, because the values of s instead of being numbers are normal vectors. We shall abstain from introducing a name for such a function, because the indicator will be the only function of this kind which we shall have to consider.

The function $f(B_A)$ is considered in a flat space—the tangent n-E.S.; its second differential is, therefore, symmetric in the independent differentials (comp. 4.3). The same applies to the indicator which, at each point, is a particular value of $f''(B_A, h_A, k_A)$. We have, thus,

15.5 $$s(h_A, k_A) = s(k_A, h_A).$$

§ 16. *Another way of introducing the indicator. Inclination.*

The aim of this section is to prove a formula which will be a generalization of 7.5; but it is obvious that we shall have to change 7.5 considerably: in fact, the left-hand side of 7.5 cannot be generalized immediately, because there is no unique normal vector at each point; the right-hand side also would

* The Vector Second Form which Wilson and Moore consider in the case $n = 2$, $N > 3$ is closely related to the so defined indicator. See § 25 of their paper in the 52 vol. of the *Proc. of the Amer. Acad. of Arts and Sciences.*

give us some difficulty; we should not be surprised therefore if the formula at which we arrive here will look altogether different from 7.5.

We introduced in § 9 the vector $x_{A \cdot B}$ as a vector of the bundle B, whose projection on the bundle A is the vector x_A. We shall adopt the same notation here.

Let us consider the expression 15.1 and 15.2 which give us the point B of Σ and the point of the n-E.S. tangent at B whose projection on the n-E.S. tangent at A is $B_A + h_A$. The difference between these two expressions

$$h_A + f'(B_A, h_A)$$

is a vector of the bundle B, whose projection on the bundle A is h_A; according to the notation introduced in this section we may therefore write

16.1 $$h_{A \cdot B} - h_A = f'(B_A, h_A).$$

Making $B_A = A + \lambda k_A$, and remembering that $f'(A, h_A) = 0$, (15.3) we have

16.2 $$\lim_{\lambda \to 0} \frac{h_{A \cdot B} - h_A}{\lambda} = f''(A, h_A, k_A) = s(h_A, k_A).$$

This formula gives us another definition of the indicator (and stands here for 7.5; it may also be compared with 7.31 and 10.1). But if we had first introduced the indicator in this way, we would have to prove its symmetry separately (given by 15.5).

The formula 16.2 is very important as it is, but we can replace it by a more general formula, removing several restrictions to which the left-hand side is subjected in its present form.

1. We defined $h_{A \cdot B}$ by its property of having h_A for its projection on the n-E.S. tangent at A; we can therefore say that the difference $h_{A \cdot B} - h_A$ is normal to Σ at A. But it is evident that it is not necessary for the validity of 16.2 to have this difference normal to A; if we would know about a vector h_B only that the ratio

16.3 $$\frac{h_B - h_A}{\lambda}$$

tends toward a vector normal to Σ at A we, evidently would have the same result.

2. Moreover, it is not necessary to have $B_A = A + \lambda k_A$; all we need is

16.4 $$\lim_{\lambda \to 0} \frac{B - A}{\lambda} = k_A$$

3. Finally, instead of considering 16.3 we may consider

$$\lim_{B\to A} \frac{h_B - h_A}{|B - A|}$$

where $|B - A|$ denotes the length of the vector $B - A$; this differs from the limit of 16.3 only by a constant factor, viz. $|k_A|$, when 16.4 holds.

We shall now sum up these remarks in the following form.

Definition. A variable vector h_B is said to *incline* toward a constant vector h_A if

$$\lim_{B\to A} \frac{h_B - h_A}{|B - A|}$$

is normal to Σ at the point A.

Theorem. If $\dfrac{B - A}{\lambda} \to k_A$ when $\lambda \to 0$ and h_B inclines toward h_A

16.5 $$\lim_{\lambda\to 0} \frac{h_B - h_A}{\lambda} = s(h_A, k_A)$$

where $s(h_A, k_A)$ is the indicator.

We have thus a new process of "inclination" which, as might be seen from § 9, and will be also seen later, plays a fundamental rôle in differentiation. This process of inclination is analogous in some respects to the process of tending toward a limit. We also have here a constant vector h_A and a variable vector h_B (belonging to a variable bundle); if we have two vectors h_B and k_B which incline toward h_A and k_A respectively, their sum $h_B + k_B$ inclines toward $h_A + k_A$ etc.

The vector $h_{A\cdot B}$ introduced in the beginning of this section furnishes us with an illustration of a vector which inclines toward h_A; another vector which inclines toward A is $h_{A.B}$ with the dot below: *we define it as the projection of* h_A on the bundle B.

We notice for further use that the relations 9.6 and 9.7 hold also in the general case; 9.7 is an immediate consequence of the definitions of $x_{A\cdot B}$ and $x_{A.B}$ and the proof of 9.6 is also easy.

Formula 16.5 contains in a nut-shell the whole general theory of curved spaces as far as second order properties are concerned.

Remark. We shall have to use in § 20 a still more general form of it: instead of considering in the left-hand side the constant vector h_A we may introduce there a variable vector $\bar{h}_A$ of the bundle A having h_A for its limit.

§ 17. *Internal and external properties of a curved space.*

The Riemann Tensor.

The indicator is essentially something external with respect to the space Σ, since it is a function whose values are *normal* vectors. The tangential vectors and inclination may, on the contrary, be interpreted in terms of the space itself, or in terms accessible to the inhabitants of the space. As concerns, first, vectors, formula 16.4 shows how a tangential vector can be introduced with the aid of a fixed point A, a variable point B and a variable number λ. This does not seem to be a very simple way of introducing vectors, but it is, as far as I can see, the simplest (if one does not introduce geodesics first; but the notion of geodesics is evidently more complicated than that of vectors) and appears to be quite natural after one thinks of it for a while. Once the vectors are introduced, inclination, which is a relation amongst vectors, also can be introduced—though not defined (except by its properties), because the definition of inclination involves subtraction of vectors belonging to two different bundles which operation cannot be performed by the inhabitants of Σ; given a variable vector x_B, they cannot decide by themselves whether it inclines toward x_A unless they are told; but we assume that inclination *is* introduced; i. e. the inhabitants *are* told of each variable vector whether it inclines toward a fixed vector.

We consider thus as internal the *vectors* of different bundles, the *operations* (addition, multiplication) on vectors belonging to the same bundle, and the notion of *inclination* (inclination is not independent of addition; but we shall not insist on this point here). All the properties which can be expressed in terms of these notions we shall call internal properties—all the others—external.

The question arises now whether all the properties of the indicator are external properties or whether we can learn something about it from the internal properties—from the inclination. We shall see in fact that some of the properties of the indicator are internal properties.

We start with the identity

$$17.1 \qquad x_{A'B.C} \cdot y_{A'C} - x_{A'C.B} \cdot y_{A'B} = \begin{vmatrix} x_{A'B} - x_A & y_{A'B} - y_A \\ y_{A'C} - y_A & y_{A'C} - y_A \end{vmatrix} .$$

Since $x_{A'B.C}$ is the projection of $x_{A'B}$ on the n-E.S. tangent to Σ at C which contains $y_{A'C}$, we may apply 14.1 and write $x_{A'B} \cdot y_{A'C}$ instead of the first term on the left-hand side. Treating in the same way the second term and writing

out the determinant we may write the identity to be proved as

$$(x_{A'B} - x_A)(y_{A'C} - y_A) - (x_{A'C} - x_A)(y_{A'B} - y_A) \\ - x_{A'B}y_{A'C} + x_{A'C}y_{A'B} = 0.$$

But instead of this we can write

$$-(x_{A'B} - x_A).y_A + (x_{A'C} - x_A).y_A - (y_{A'C} - y_A).x_A \\ + (y_{A'B} - y_A).x_A = 0;$$

and this is true because in each of the products the first factor is perpendicular to the bundle A (compare 15.1) and, therefore, to the second factor.

If now we make

$$\lim_{\lambda\to 0} \frac{B - A}{\lambda} = h_A \qquad \lim_{\mu\to 0} \frac{C - A}{\mu} = k_A$$

and take account of 17.1 and 16.2 we find

$$17.2 \qquad \lim_{\substack{\lambda\to 0\\ \mu\to 0}} \frac{x_{A'B.C}.y_{A'C} - x_{A'C.B}.y_{A.B}}{\lambda\mu} = \begin{vmatrix} s(x_A, h_A) & s(x_A, k_A) \\ s(y_A, h_A) & s(y_A, k_A) \end{vmatrix}.$$

The left-hand side is an expression which does not involve operations on vectors belonging to different bundles. It is connected therefore with inner properties of the space, and the right-hand side involves the indicator; more precisely, if we know the indicator we can calculate the right-hand side but not vice versa, of course; the determinant of the right-hand side reflects only *some* of the properties of the indicator—we can change the indicator to a certain degree without changing this determinant—and we see that *these* properties of the indicator are internal properties of the space.

Just as in the preceding section, it is possible to generalize the formula 17.2 by removing some restrictions from the left-hand side; viz., we write our formula as

$$17.3 \qquad \lim_{\substack{\lambda\to 0\\ \mu\to 0}} \frac{x_{ABC}.y_{AC} - x_{ACB}.y_{AB}}{\lambda\mu} = \begin{vmatrix} s(x_A, h_A) & s(x_A, k_A) \\ s(y_A, h_A) & s(y_A, k_A) \end{vmatrix}.$$

Here x_{AB} denotes a vector,—function of x_A and B, which inclines toward x_A; x_{ABC} a vector, function of x_{AB} and C, which inclines toward x_{AC} etc. It is thus enough only to know the inclination in order to be able to compute the determinant of the right-hand side. This determinant is a function which has numerical values and which depends linearly on four vector arguments x, y, h, k. It is therefore a tensor, and this is precisely the Riemann Tensor of Σ at the point A. We denote it by $\rho(x, y; h, k)$. Its fundamental properties, i. e. the fact that it changes its sign when we interchange x and y, or h and k, the fact that it does not change when we interchange the couples

x, y and h, k and the relation

$$\rho(x, y; h, k) + \rho(x, h; k, y) + \rho(x, k; y, h) = 0,$$

are immediate consequences of its form 17.3 and of the symmetry of the indicator (15.5).

§ 18. *Tensorfields of a curved space and their differentiation.*

As in § 5, we consider a tensor, whose arguments are vectors of a bundle tangent to Σ, as a tensor belonging to Σ; and a *tensorfield* belonging to Σ will be constituted by tensors given in each bundle of Σ, or, what is the same thing, by a function $\phi(A, x_A, y_A, \cdots)$ with one point argument, which is a point of Σ, and several vector arguments, which are vectors of the corresponding bundle and upon which the function depends linearly.

As an example we may again consider the "fundamental" tensorfield

18.1 $$\epsilon(A, x_A, y_A) = x_A \cdot y_A,$$

which, however, is inconspicuous in the theory unless we introduce the usual coördinate representation of curved space. As another example may serve the tensorfield constituted by the Riemann Tensors at the points of Σ. The indicator does *not* constitute a tensorfield here, since its values are not numbers.

Using the notion of *inclination*, we can define the differential of a tensorfield $\phi(A, x_A, y_A, \cdots)$, in accordance with § 9, by

18.2 $$\phi'(A, x_A, y_A, \cdots, h_A) = \lim_{\lambda \to 0} \frac{\phi(B, x_B, y_B, \cdots) - \phi(A, x_A, y_A, \cdots)}{\lambda},$$

with $\dfrac{B - A}{\lambda} \to h_A$, and x_B, y_B, $\cdots$ being vectors which *incline* toward x_A, y_A, $\cdots$, respectively. In particular we can put $x_B = x_{A \cdot B}$ or $x_B = x_{A,B}$ etc.

If we consider again the fundamental tensor 18.1, we can prove that its differential is zero. In fact

$$\epsilon'(x_A, y_A, h_A) = \lim_{\lambda \to 0} \frac{x_{AB} \cdot y_{AB} - x_A \cdot y_A}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{x_B - x_A}{\lambda} \cdot y_B - \lim_{\lambda \to 0} x_A \cdot \frac{y_A - y_B}{\lambda},$$

and this is zero because $\lim \dfrac{x_B - x_A}{\lambda}$ and $\lim \dfrac{y_B - y_A}{\lambda}$ are normal vectors according to the definition of inclination, and are, therefore, perpendicular to x_A and $\lim y_B$.

We pass now to the generalization of the Codazzi equations (§ 10). We

remember that the *indicator* is equal to the *second differential* of $f(B_A)$ at the point A; the relation 10.6 from which we deduced the Codazzi equations 10.7 in the case $n = 2$ can be said to express a connection between the *differential of the indicator* (which is connected with the space Σ, so that the differential is taken "along" Σ) and the *third differential of* $f(B_A)$ (which is taken along the tangent n-E.S.). We proceed now to find out how s' and f''' are connected in the general case.

We make $B_A = A + \mu l_A$, $C_A = B_A + \lambda k_A$, and have

$$\lim_{\lambda \to 0} \frac{C - B}{\lambda} = k_{A \cdot B},$$

as in § 10.

According to 16.1, we have

$$h_{A \cdot B} - h_A = f'(B_A, h_A), \qquad h_{A \cdot C} - h_A = f'(C_A, h_A);$$

so that

$$f''(B_A, h_A, k_A) = \lim_{\lambda \to 0} \frac{h_{A \cdot C} - h_{A \cdot B}}{\lambda}.$$

On the other hand, according to 16.2;

$$s(h_{A \cdot B}, k_{A \cdot B}) = \lim_{\lambda \to 0} \frac{h_{A \cdot B \cdot C} - h_{A \cdot B}}{\lambda},$$

Since, for $B = A$, the values of s and f'' coincide (by definition) we have

18.3 $$s'(h_A, k_A, l_A) - f'''(A_A, h_A, k_A, l_A) = \lim_{\substack{\lambda \to 0 \\ \mu \to 0}} \frac{h_{A \cdot B \cdot C} - h_{A \cdot C}}{\lambda \mu}.$$

From this it follows at once that *the normal component of* s' *is equal to* f''', which is the sought-for generalization of 10.6; and from this we arrive at the consequence, which may be looked upon as the generalization of the Codazzi relations, that

18.4 *the normal component of the differential of the indicator is symmetrical in its three arguments.*

Incidentally, if we want to consider the tangential component of the difference 18.3 we write the right-hand side with the dots below (which does not change its value) and multiply it by an arbitrary vector z_A. We have $h_{A.B.C} \cdot z_A = h_{A.B} z_{A.C}$, $h_{A.C} \cdot z_A = h_A z_{A.C}$, and the difference takes the form

$$\lim_{\substack{\lambda \to 0 \\ \mu \to 0}} \frac{h_{A.B} - h_A}{\mu \lambda} \cdot z_{A.C}.$$

We first pass to the limit $\mu \to 0$ and obtain

$$\lim_{\lambda \to 0} \frac{s(h_A, l_A) \,.\, z_{A.C}}{\lambda} \,.$$

Since $s(h_A, l_A)$ is perpendicular to z_A, we write this

$$\lim_{\lambda \to 0} \left(s(h_A, l_A) \,.\, \frac{z_{A.C} - z_A}{\lambda} \right) \,,$$

which gives $s(h_A, l_A) \,.\, s(z_A, k_A)$. We thus have, dropping the subscript A,

$$z \,.\, s'(h, k, l) = s(h, l) \,.\, s(z, k).$$

If we interchange k and l and take the difference, we obtain:

$$z \,.\, s'(h, k, l) - z \,.\, s'(h, l, k) = \begin{vmatrix} s(h, l) & s(h, k) \\ s(z, l) & s(z, k) \end{vmatrix} = \rho(h, z; l, k) \,.$$

The situation is then, the tangential component of the differential of the indicator is not, in general, symmetrical in its arguments: if we interchange the two last arguments, the difference is the Riemann Tensor.

The consideration of the tangential component of s' is, however, not necessary for our purpose, which is now the consideration of the differential of the Riemann Tensor. From its definition we have

$$\rho'(x, y; h, k, l) = \begin{vmatrix} s'(x, h, l) & s'(x, k, l) \\ s(y, h) & s(y, k) \end{vmatrix} + \begin{vmatrix} s(x, h) & s(x, k) \\ s'(y, h, l) & s'(y, k, l) \end{vmatrix} \,;$$

since s' is, in each instance, multiplied by s, which is a normal vector, we need not take into account the tangential component, i. e. we can substitute for s' its normal component which, according to the generalization of the Codazzi relations 18.4, is symmetrical in its arguments. This leads to an immediate proof of the Bianchi relations 12.5 for the general case.

Tensorfields of Σ, their differentiation, and the Bianchi identities are internal things; but here they were discussed with the aid of external things like the indicator, its differential, and the Codazzi relations.

§ 19. *An r-S. in an n-S.*

In this section and the following we shall consider the generalizations of the developments of §§ 8 and 11; instead of a curve on a surface which is a 1-S. in a 2-S. we shall consider an r-S., which we shall call Σ_r, in an n-S. (with $r < n$), which we shall call Σ_n.

The question which we shall try to answer first is: what relation exists between the indicators $s_r(h_A, k_A)$ of Σ_r and $s_n(h_A, k_A)$ of Σ_n in a point A which belongs to both? First, we notice that the tangent r-E.S. of Σ_r at that point lies entirely in the n-E.S. tangent to Σ_n; we therefore shall consider in $s_n(h_A, k_A)$ only those values of h_A, k_A which belong to the tangent r-E.S. and so have the functions s_r and s_n for the same arguments.

We now introduce the formulas

19.1 $$s_n(h_A, k_A) = \lim_{\lambda \to 0} \frac{h_{AnB} - h_A}{\lambda}$$

19.2 $$s_r(h_A, k_A) = \lim_{\lambda \to 0} \frac{h_{ArB} - h_A}{\lambda}$$

where h_{AnB} and h_{ArB} denote vectors of the bundles B which incline toward h_A with respect to Σ_n and to Σ_r respectively; h_{AnB} may be the projection of h_A on the n-E.S. tangent to Σ_n at B, and h_{ArB} the projection of h_A on the r-E.S. tangent to Σ_r at the same point. Since this r-E.S. forms part of that n-E.S. the difference between these two vectors will be a vector of the n-E.S. tangent at B and perpendicular to the r-E.S. tangent at B; the difference between the limits will be a vector of the n-E.S. tangent at A and perpendicular to the r-E.S. tangent to A; we can say

The values of $s_n(h_A, k_A)$ for the arguments h_A and k_A for which $s_r(h_A, k_A)$ are determined (i. e. for h_A, k_A belonging to the r-E.S. tangent at A) *are projections of the corresponding values of $s_r(h_A, k_A)$.*

The formula 8.4 which contains the Euler and the Meusnier theorems may be interpreted as a special case of this. In fact, if we make $n = 2$, we have

19.3 $$s_2(h_A, k_A) = a.\sigma(h_A, k_A).$$

The only vectors for which $s_1(h_A, k_A)$ is determined are vectors of the tangent to our curve; since they all have the same direction, there is no loss of generality if we put $h_A = k_A = t$, where t is the unit vector of that direction. It is clear that, in this case, we have $s_1(t, t) = \dot{t}$, and the underlined proposition above takes the form

$$a(a\dot{t}) = a.\sigma(t, t)$$

or, if we take 8.1 into consideration

$$a\,\frac{(an)}{\rho} = a.\sigma(t, t).$$

Incidentally, we see what the meaning of the indicator for a 1-S., i. e. for a curve, is. In this case all the values of the indicator differ only by numerical factors of its value for unit vectors as arguments, and this last value is the curvature vector, or a vector which has the direction of the first normal and whose length is equal to the curvature of the curve. We may remark that our general point of view means a simplification even in the simplest case: the curvature is introduced without the notion of length of arc.

§ 20. *The geodesic indicator.*

We shall now generalize the operation considered in § 11. We again consider Σ_r in Σ_n. We had (19.2) for the inclination in Σ_r

$$\lim_{\lambda\to 0} \frac{h_{ArB} - h_A}{\lambda} = s_r(h_A, k_A).$$

The subtraction considered here cannot be performed in Σ_r, and is supposed to be performed in the containing N-dimensional Euclidean Space. But if we consider the n-S. Σ_n, of which Σ_r is a part and do not wish to step out of this Σ_n then, in analogy to 11.2 we shall consider

$$\lim \frac{h_{ArBnA} - h_A}{\lambda}$$

as the indicator of Σ_r in Σ_n, or as the *geodesic indicator* of Σ_r in Σ_n. But we have

$$\lim \frac{h_{ArBnA} - h_A}{\lambda} = \lim \frac{h_{ArBnA} - h_{ArB} + h_{ArB} - h_A}{\lambda}$$

$$= \lim_{\lambda\to 0} \frac{h_{ArB} - h_A}{\lambda} - \lim_{\lambda\to 0} \frac{h_{ArB} - h_{ArBnA}}{\lambda}$$

The minuend of the last expression is the indicator of Σ_r, and the subtrahend—as the remark at the end of § 16 permits to conclude—is the indicator of Σ_n. Hence, the geodesic indicator of Σ_r in Σ_n is the difference between the indicator of Σ_r (in the containing N-dimensional Euclidean Space) and the indicator of Σ_n.

Or, we may say also: if the indicator of Σ_r is decomposed into its tangential and normal components with respect to Σ_n, in which Σ_r lies, the tangential component is the geodesic indicator of Σ_r in Σ_n, and the values of the normal component are equal to the corresponding values of the indicator of Σ_n.

The results of § 11 are easily seen to follow from this as a special case.

These considerations may easily be extended. If Σ_t lies in Σ_r, Σ_r in Σ_n, the geodesic indicator of Σ_t with respect to Σ_n is equal to the geodesic indi-

cator of Σ_t with respect to Σ_r plus the geodesic indicator of Σ_r with respect to Σ_n.

AUGUST 18, 1924.

ADDED NOV. 15, 1924.

There exists a more immediate generalization of the formula 7.5 of the first paper than that given by 16.2. In § 7 we considered at every point B, given by $B_A = A + \lambda k_A$, *the* unit normal vector b, a being the unit normal vector at A. We cannot speak here of *the* unit normal vector; but we shall consider at each point B some definite normal vector b, a being the vector which corresponds to $\lambda = 0$. The

$$\lim \frac{b-a}{\lambda},$$

when this limit exists, is not, in general, a vector of the tangent bundle A; we therefore consider the tangential component of $\lim \frac{b-a}{\lambda}$; this tangential component is a vector which depends only on a and k_A. We shall denote it by $\mathcal{F}(a, k_A)$. It can be shown that $\mathcal{F}$ depends on each argument linearly, and that the connection of this function $\mathcal{F}(a, k_A)$ with the indicator $s(h, k)$ of § 15 is given by the relation

$$h_A . \mathcal{F}(a, k_A) = a . s(h, k).$$

Here h_A and k_A are two vectors of the tangent bundle at A, and a a vector of the normal bundle.

Errata in the first paper:

P. 83, line 2 from above: read k_A instead of h_A.
P. 86, line 1 from above: read $x_{A'B}$ instead of $x_{B'A}$.
P. 88, line 6 from above: read $C_A = B_A + \lambda k_A$ instead of $C_A = B + \lambda k_A$.
P. 89, line 9 from above: read δ'' instead of δ.
P. 92, line 9 from above, read $x_{A.B.C.A} - x_{A.C.B.A}$ instead of $x_{A.B.C.A} - x_{A.B.C.A}$

Generalization of Certain Theorems of Bohl.

By F. H. Murray.

It is the object of this paper to generalize a certain number of theorems of P. Bohl * concerning the trajectories of a mechanical system in the neighborhood of a point of equilibrium. The coördinates of the system considered by Bohl satisfied equations of the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega'_i}\right) - \frac{\partial T}{\partial \omega_i} = \frac{\partial U}{\partial \omega_i}, \qquad i = 1, \cdots, n,$$

in which T is a definite quadratic form in the velocities, and the terms of the second order in U are a non-singular quadratic form. Under these and certain other assumptions the existence is established of trajectories which remain in a given neighborhood of the point of equilibrium for $t \geq t_0$, for $t \leq t_0$, or for all values of t; and the manifold of each type of trajectory is determined.

In the present paper a number of similar theorems are obtained concerning the trajectories in the neighborhood of a point solution of the equations of Hamilton

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \qquad i = 1, \cdots, n,$$

in which F does not contain the time. A certain number of inequalities are also obtained for trajectories in the neighborhood of a periodic orbit, when F contains t explicitly. While it seems probable that the results obtained here can be still further generalized, they are sufficient for a number of applications which will be made in a following paper.

Although the theorems obtained include those of Bohl as a special case, no claim to originality of treatment is made, since his methods, with slight changes, are sufficient for this more general problem. A new difficulty which occurs here is that of determining the form of F after certain transformations; this is overcome by a method similar to one employed by Poincaré.†

* P. Bohl, "Über die Bewegung eines mechanischen Systems," *Journal für die reine und angewandte Mathematik*, Band 127 (1904), pp. 179-276.

† *Les méthodes nouvelles de la mécanique céleste*, t. III, § 319.

PART I. *Transformation of the equations of motion.*

1. The equations of variation.

Suppose the motion of the system defined by the canonical equations

$$(1) \qquad \frac{d\bar{x}_i}{dt} = \frac{\partial F}{\partial \bar{y}_i}, \quad \frac{d\bar{y}_i}{dt} = -\frac{\partial F}{\partial \bar{x}_i}, \qquad i = 1, \cdots, n,$$

and let a periodic solution be represented by the functions

$$(2) \qquad \bar{x}_i = \phi_i(t), \quad \bar{y}_i = \psi_i(t) .$$

Suppose the region (D) to consist of all points $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n, \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n)$ which satisfy the inequalities

$$(D) \qquad |\bar{x}_i - \phi_i(t)| < d', \quad |\bar{y}_i - \psi_i(t)| < d', \qquad i = 1, \cdots, n$$

for some value of t; it will be assumed that the function F and its partial derivatives of the first and second orders are continuous within this region. If the transformation $\bar{x}_i = \phi_i + \xi_i$, $\bar{y}_i = \psi_i + \eta_i$ is made, from (1) we obtain the equations

$$(3) \qquad d\xi_i/dt = [\partial F/\partial \bar{y}] - [\partial F/\partial \bar{y}_i]^0, \; d\eta_i/dt = -[\partial F/\partial \bar{x}_i] + [\partial F/\partial \bar{x}_i]^0$$

or if

$$\bar{F} = [F] - \sum_{i=1}^{n} ([\partial F/\partial \bar{x}_i]^0 \xi_i + [\partial F/\partial \bar{y}_i]^0 \eta_i) - F^0$$

$$(4) \qquad d\xi_i/dt = \partial \bar{F}/\partial \eta_i, \qquad d\eta_i/dt = -\partial \bar{F}/\partial \xi_i, \qquad i = 1, \cdots, n.$$

The equations of variation for this system have the form

$$(4') \qquad \begin{aligned} \frac{d\bar{\xi}_i}{dt} &= \sum_{k=1}^{n} \frac{\partial^2 F}{\partial \bar{y}_i \partial \bar{x}_k} \bar{\xi}_k + \sum_{k=1}^{n} \frac{\partial^2 F}{\partial \bar{y}_i \partial \bar{y}_k} \bar{\eta}_k \\ \frac{d\bar{\eta}_i}{dt} &= -\sum_{k=1}^{n} \frac{\partial^2 F}{\partial \bar{x}_i \partial \bar{x}_k} \bar{\xi}_k - \sum_{k=1}^{n} \frac{\partial^2 F}{\partial \bar{x}_i \partial \bar{y}_k} \bar{\eta}_k \end{aligned} \qquad i = 1, 2, \cdots, n$$

the coefficients being periodic functions of t, of the same period as the functions (2). If this period is T, and the functions (2) are not constants, it is well known that the characteristic exponents can be obtained by the construction of a fundamental set of solutions of (4'), $(\bar{\xi}_i^{(r)}, \bar{\eta}_i^{(r)})$, $(r = 1, \cdots, 2n)$ such that for $t = 0$,

$$\begin{aligned} &\bar{\xi}_s^{(r)} = 0, \quad r \neq s, \qquad \bar{\xi}_r^{(r)} = 1, \\ &\bar{\eta}_s^{(n+r)} = 0, \quad r \neq s, \quad \bar{\eta}_r^{(n+r)} = 1, \qquad r, s = 1, \cdots, n. \end{aligned}$$

Then if s is a root of the equation

$$\Delta(s) = \begin{vmatrix} \bar{\xi}'_1(T)-s & \bar{\xi}''_1(T) & \cdots & \bar{\xi}_1^{(n)}(T) & \bar{\xi}_1^{(n+1)}(T) & \cdots & \bar{\xi}_1^{(2n)}(T) \\ \bar{\xi}'_2(T) & \bar{\xi}''_2(T)-s & \cdots & \bar{\xi}_2^{(n)}(T) & \bar{\xi}_2^{(n+1)}(T) & \cdots & \bar{\xi}_2^{(2n)}(T) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{\xi}'_n(T) & \bar{\xi}''_n(T) & \cdots & \bar{\xi}_n^{(n)}(T)-s & \bar{\xi}_n^{(n+1)}(T) & \cdots & \bar{\xi}_n^{(2n)}(T) \\ \bar{\eta}'_1(T) & \bar{\eta}''_1(T) & \cdots & \bar{\eta}_1^{(n)}(T) & \bar{\eta}_1^{(n+1)}(T)-s & \cdots & \bar{\eta}_1^{(2n)}(T) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{\eta}'_n(T) & \bar{\eta}''_n(T) & \cdots & \bar{\eta}_n^{(n)}(T) & \bar{\eta}_n^{(n+1)}(T) & \cdots & \bar{\eta}_n^{(2n)}(T)-s \end{vmatrix} = 0$$

the period T can be so chosen that every real root is positive.* The exponent corresponding to a root s is determined from the equation

$$s = e^{aT}$$

and these exponents occur in pairs † $(a, -a)$. The exponents corresponding to conjugate imaginary roots of $\Delta(s) = 0$ can evidently be grouped in conjugate pairs, if F is a real function of its arguments.

In the developments which follow it will be assumed that the elementary divisors of the matrix formed from the elements of the determinant above are linear; under this hypothesis, and the further assumption that no exponent is congruent to zero, (mod $2\pi i/T$) a fundamental set of solutions of (4′) can be represented in the form

$$\begin{aligned} &\bar{\xi}_i^{(r)} = e^{a_r t}\xi_i^{(r)}, \quad \bar{\eta}_i^{(r)} = e^{a_r t}\eta_i^{(r)}, \qquad i = 1, \cdots, n, \\ &\bar{\xi}_i^{(n+r)} = e^{-a_r t}\xi_i^{(n+r)}, \quad \bar{\eta}_i^{(n+r)} = e^{-a_r t}\eta_i^{(n+r)}, \qquad r = 1, \cdots, n, \end{aligned} \tag{5}$$

in which the functions $\xi_i^{(s)}$, $\eta_i^{(s)}$ are periodic functions, of period T.

If the functions ϕ_i, ψ_i are constants, the exponents are defined as the roots of the equation

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial\bar{y}_1\partial\bar{x}_1}-a, & \cdots & \dfrac{\partial^2 F}{\partial\bar{y}_1\partial\bar{x}_n} & \dfrac{\partial^2 F}{\partial\bar{y}_1\partial\bar{y}_1} & \cdots & \dfrac{\partial^2 F}{\partial\bar{y}_1\partial\bar{y}_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 F}{\partial\bar{y}_n\partial\bar{x}_1} & \cdots & \dfrac{\partial^2 F}{\partial\bar{y}_n\partial\bar{x}_n}-a, & \dfrac{\partial^2 F}{\partial\bar{y}_n\partial\bar{y}_1} & \cdots & \dfrac{\partial^2 F}{\partial\bar{y}_n\partial\bar{y}_n} \\ -\dfrac{\partial^2 F}{\partial\bar{x}_1\partial\bar{x}_1} & \cdots & -\dfrac{\partial^2 F}{\partial\bar{x}_1\partial\bar{x}_n} & -\dfrac{\partial^2 F}{\partial\bar{x}_1\partial\bar{y}_1}-a, & \cdots & -\dfrac{\partial^2 F}{\partial\bar{x}_1\partial\bar{y}_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\dfrac{\partial^2 F}{\partial\bar{x}_n\partial\bar{x}_1} & \cdots & -\dfrac{\partial^2 F}{\partial\bar{x}_n\partial\bar{x}_n} & -\dfrac{\partial^2 F}{\partial\bar{x}_n\partial\bar{y}_1} & \cdots & -\dfrac{\partial^2 F}{\partial\bar{x}_n\partial\bar{y}_n}-a \end{vmatrix} = 0.$$

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* Goursat-Hedrick, *Mathematical Analysis*, Vol. II, Part II, p. 149, also pp. 164-167.

† Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, tome 1, p. 196.

It will be assumed that the elementary divisors of the matrix corresponding to this determinant are linear, and that no root is zero. With these hypotheses the solutions of (4′) are in the form (5), the functions $(\xi_i^{(s)}, \eta_i^{(s)})$ being constants.

From the functions of (5) can be constructed a *normal* set possessing the following properties: if the real part of a_r is zero, the functions $\xi_i^{(r)}$, $\xi_i^{(n+r)}$ are conjugate imaginaries, as are also $\eta_i^{(r)}$, $\eta_i^{(n+r)}$; either the real part of a_r is positive, or $a_r/\sqrt{-1}$ is positive, $r = 1, \cdots, n$; if a_r and a_s are conjugates, $\xi_i^{(r)}$, $\xi_i^{(s)}$ are conjugates, as are also $\eta_i^{(r)}$, $\eta_i^{(s)}$; and finally,

$$(6) \qquad \sum_{i=1}^{n} (\xi_i^{(r)}\eta_i^{(s)} - \eta_i^{(r)}\xi_i^{(s)}) = 0, \qquad s \neq n + r \text{ and } r \neq n + s$$
$$= \epsilon_r \neq 0, \quad s = n + r, \quad r = 1, \cdots, n$$

where $\epsilon_r = 1/a_r$ if a_r is real, $\epsilon_r = 1/2a_r$ if a_r is neither real nor pure imaginary, and $\epsilon_r = \pm 1/a_r$ if a_r is a pure imaginary.

For, as a result of the relation between any pair of solutions of (4′), we obtain

$$\sum_{i=1}^{n} (\bar{\xi}_i^{(r)}\bar{\eta}_i^{(s)} - \bar{\xi}_i^{(r)}\bar{\eta}_i^{(s)}) = C$$

$$\sum_{i=1}^{n} (\xi_i^{(r)}\eta_i^{(s)} - \eta_i^{(r)}\xi_i^{(s)}) = 0, \quad a_r + a_s \neq 0.$$

The exponents can evidently be chosen as required, and the functions $\xi_i^{(s)}$, $\eta_i^{(s)}$ can be grouped in the required manner without loss of generality. It remains to satisfy (6). From the investigations of Liapounoff and others it is known that if there are m sets corresponding to a_r, there are m corresponding to $-a_r$; if the corresponding sets of functions are $(\xi_i^{(r)}, \eta_i^{(r)})$, $(\xi_i^{(n+r)}, \eta_i^{(n+r)})$, $(r = 1, \cdots, m)$, suppose first that the real part of a_r is not zero. The functions of (5) being linearly independent, some k must exist for which

$$\sum_{i=1}^{n} (\xi'_i\eta_i^{(n+k)} - \eta'_i\xi_i^{(n+k)}) \neq 0, \quad 1 \leq k \leq m$$

and for convenience suppose $k = 1$. If functions

$$\xi_i = \sum_{r=1}^{m} C_r\xi_i^{(n+r)}, \quad \eta_i = \sum_{r=1}^{m} C_r\eta_i^{(n+r)}, \qquad i = 1, \cdots, n$$

are constructed, the constants $C_1, \cdots, C_m$ can be so chosen that

$$\sum_{i=1}^{n} (\xi'_i\eta_i - \eta'_i\xi_i) = \sum_{r=1}^{m} C_r \sum_{i=1}^{n} (\xi'_i\eta_i^{(n+r)} - \eta'_i\xi_i^{(n+r)}) = 0.$$

The coefficient of C_1 being different from zero, this equation has $(m - 1)$

independent solutions, $(C_1^{(r)}, C_2^{(r)}, \cdots, C_m^{(r)})$ and the corresponding sets of functions $(\bar{\xi}_i^{(n+r)}, \bar{\eta}_i^{(n+r)})$ are independent sets.*

Similarly if †

$$\bar{\xi}_i = \sum_{r=1}^{m} C_r \xi_i^{(r)}, \quad \bar{\eta}_i = \sum_{r=1}^{m} C_r \eta_i^{(r)} \qquad i = 1, \cdots, n$$

the equation

$$\sum_{i=1}^{n} (\xi_i^{(n+1)} \bar{\eta}_i - \eta_i^{(n+1)} \bar{\xi}_i) = \sum_{r=1}^{m} C_r \sum_{i=1}^{n} (\xi_i^{(n+1)} \eta_i^{(r)} - \eta_i^{(n+1)} \xi_i^{(r)}) = 0$$

has $(m-1)$ independent solutions, and the sets of functions $(\bar{\xi}_i^{(s)}, \bar{\eta}_i^{(s)})$, $(s = 2, \cdots, m)$ are linearly independent. With the $2(m-1)$ sets thus obtained one may proceed in the same manner, until a set has been obtained for which

$$\sum_{i=1}^{n} (\xi_i^{(r)} \eta_i^{(s)} - \eta_i^{(r)} \xi_i^{(s)}) = 0, \quad s \neq n + r,$$
$$= a_r \neq 0, \quad s = n + r, \quad r = 1, \cdots, m;$$

if the functions of each set $(\xi_i^{(r)}, \eta_i^{(r)})$ are multiplied by a suitable constant, the equations

$$\sum_{i=1}^{n} (\xi_i^{(r)} \eta_i^{(n+r)} - \eta_i^{(r)} \xi_i^{(n+r)}) = 1/a_r, \ a_r \text{ real},$$
$$= 1/2a_r, \ R(a_r/i) \neq 0, \ r = 1, \cdots, m$$

are obtained.

If a_r is a pure imaginary, the corresponding functions of the sets $(\xi_i^{(r)}, \eta_i^{(r)})$, $(\xi_i^{(n+r)}, \eta_i^{(n+r)})$ may be assumed conjugates; if

$$\sum_{i=1}^{n} (\xi_i^{(r)} \eta_i^{(n+r)} - \eta_i^{(r)} \xi_i^{(n+r)}) = 0, \quad r = 1, \cdots, m,$$

then

$$\sum_{i=1}^{n} (\xi'_i \eta_i^{(n+s)} - \eta'_i \xi_i^{(n+s)}) = a\sqrt{-1}, \ a \neq 0, \ 1 < s \leq m.$$

and taking conjugates, a being real,

$$\sum_{i=1}^{n} (\xi_i^{(n+1)} \eta_i^{(s)} - \eta_i^{(n+1)} \xi_i^{(s)}) = -a\sqrt{-1}.$$

Consequently if

$$\bar{\xi}'_i = \xi'_i + \xi_i^{(s)}, \quad \bar{\eta}'_i = \eta'_i + \eta_i^{(s)},$$
$$\bar{\xi}_i^{(n+1)} = \xi_i^{(n+1)} + \xi_i^{(n+s)}, \quad \bar{\eta}_i^{(n+1)} = \eta_i^{(n+1)} + \eta_i^{(n+s)},$$

then

$$\sum_{i=1}^{n} (\bar{\xi}'_i \bar{\eta}_i^{(n+1)} - \bar{\eta}'_i \bar{\xi}_i^{(n+1)}) = 2a\sqrt{-1};$$

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* Bocher, *Introduction to Higher Algebra*, (1922), p. 50.

† The dashed variables used here are not to be confused with those of (5).

and if each function of the sets $(\bar{\xi}'_i, \bar{\eta}'_i)$, $\bar{\xi}_i^{(n+1)}, \bar{\eta}^{(n+1)})$ is multiplied by the same real constant, suitably chosen, the relation

$$\sum_{i=1}^{n} (\bar{\xi}'_i \bar{\eta}_i^{(n+1)} - \bar{\eta}'_i \bar{\xi}_i^{(n+1)}) = \pm 1/a_1$$

is obtained. Hence this relation must hold for at least one pair of conjugate sets. Then as before, from the equation

$$\sum_{r=1}^{m} C_r \sum_{i=1}^{n} (\bar{\xi}'_i \eta_i^{(n+r)} - \bar{\eta}'_i \xi_i^{(n+r)}) = 0$$

and its conjugate

$$\sum_{r=1}^{m} \bar{C}_r \sum_{i=1}^{n} (\bar{\xi}_i^{(n+1)} \eta_i^{(r)} - \bar{\eta}_i^{(n+1)} \xi_i^{(r)}) = 0$$

can be determined linearly independent sets of functions $(\bar{\xi}_i^{(r)}, \bar{\eta}_i^{(r)})$, $(\bar{\xi}_i^{(n+r)}, \bar{\eta}_i^{(n+r)})$, $(r = 2, \cdots, m)$ which can be operated on as before until sets have been obtained for which

$$\sum_{i=1}^{n} (\bar{\xi}_i^{(r)} \bar{\eta}_i^{(n+s)} - \bar{\eta}_i^{(r)} \bar{\xi}_i^{(n+s)}) = 0, \; s \neq r, \quad r, s = 1, \cdots, m,$$
$$= \pm 1/a_r, \quad s = r,$$

the functions $\bar{\xi}_i^{(r)}, \bar{\xi}_i^{(n+r)}$ being conjugates, and the same is true of $\bar{\eta}_i^{(r)}, \bar{\eta}_i^{(n+r)}$.

When one or the other procedure has been applied to each exponent, conjugate functions $\xi_i^{(r)}$, $\xi_i^{(s)}$, $\eta_i^{(r)}$, $\eta_i^{(s)}$ being chosen to correspond to conjugate imaginary exponents as required, the resulting set will be a normal one; and in what follows it will be assumed that the periodic functions or constants in (5) form a normal set.

2. Reduction of the differential equations to a simple form.

Equations (4), (4′) can now be reduced to a simpler form by means of the change of variables

$$\xi_i = \sum_{r=1}^{n} \xi_i^{(r)} \bar{u}_r + \sum_{r=1}^{n} \xi_i^{(n+r)} \bar{v}_r$$
$$\eta_i = \sum_{r=1}^{n} \eta_i^{(r)} \bar{u}_r + \sum_{r=1}^{n} \eta_i^{(n+r)} \bar{v}_r \qquad i = 1, \cdots, n, \tag{7}$$

after which (4) become

$$d\bar{u}_r/dt = a_r \bar{u}_r + \phi'_r, \quad d\bar{v}_r/dt = -a_r \bar{v}_r + \psi'_r \qquad r = 1, \cdots, n. \tag{8}$$

If a is an arbitrary constant of integration, from (6), (7), we obtain

$$\frac{\partial \bar{F}}{\partial a} = \sum_{i=1}^{n} \left(\frac{d\xi_i}{dt} \frac{\partial \eta_i}{\partial a} - \frac{d\eta_i}{dt} \frac{\partial \xi_i}{\partial a} \right)$$

$$= \frac{\partial \Omega}{\partial a} + \sum_{i=1}^{n} \epsilon_r \left(\frac{d\bar{u}_r}{dt} \frac{\partial \bar{v}_r}{\partial a} - \frac{d\bar{v}_r}{dt} \frac{\partial \bar{u}_r}{\partial a} \right)$$

where

$$\Omega = \sum_{i=1}^{n} \frac{\bar{u}_r^2}{2} \sum_{i=1}^{n} \left(\frac{d\xi_i^{(r)}}{dt} \eta_i^{(r)} - \frac{d\eta_i^{(r)}}{dt} \xi_i^{(r)} \right)$$

$$+ \sum_{r<s} \bar{u}_r \bar{u}_s \sum_{i=1}^{n} \left(\frac{d\xi_i^{(r)}}{dt} \eta_i^{(s)} - \frac{d\eta_i^{(r)}}{dt} \xi_i^{(s)} \right)$$

$$+ \sum_{r=1}^{n} \frac{\bar{v}_r^2}{2} \sum_{i=1}^{n} \left(\frac{d\xi_i^{(n+r)}}{dt} \eta_i^{(n+r)} - \frac{d\eta_i^{(n+r)}}{dt} \xi_i^{(n+r)} \right)$$

$$+ \sum_{r<s} \bar{v}_r \bar{v}_s \sum_{i=1}^{n} \left(\frac{d\xi_i^{(n+r)}}{dt} \eta_i^{(n+s)} - \frac{d\eta_i^{(n+r)}}{dt} \xi_i^{(n+s)} \right)$$

$$+ \sum_{r,s=1}^{n} \bar{u}_r \bar{v}_s \sum_{i=1}^{n} \left(\frac{d\xi_i^{(r)}}{dt} \eta_i^{(n+s)} - \frac{d\eta_i^{(r)}}{dt} \xi_i^{(n+s)} \right)$$

Consequently * if $F' = \bar{F} - \Omega$, the equations in $\bar{u}_i$, $\bar{v}_i$, are in the form

$$(9) \qquad \frac{d\bar{u}_i}{dt} = \frac{\partial F'}{\partial(\epsilon_i \bar{v}_i)}, \quad \epsilon_i \frac{d\bar{v}_i}{dt} = -\frac{\partial F'}{\partial \bar{u}_i} \qquad i = 1, \cdots, n$$

and from (8),

$$(10) \qquad F' = \sum_{r=1}^{n} \epsilon_r a_r \bar{u}_r \bar{v}_r + F_3$$

If the functions $(\xi_i^{(s)}, \eta_i^{(s)})$ are constants, $\Omega = 0$; in any case

$$(11) \qquad \frac{\partial^2 F_3}{\partial \bar{u}_r \partial \bar{u}_s} = \frac{\partial^2 F_3}{\partial \bar{u}_r \partial \bar{v}_s} = \frac{\partial^2 F_3}{\partial \bar{v}_r \partial \bar{v}_s} = 0, \quad \bar{u}_r = \bar{u}_s = 0, \quad r, s, = 1, \cdots, n.$$

The function F having been defined only for real values of ξ_i, η_i, it is necessary to give $\bar{u}_i$, $\bar{v}_i$ values for which the original variables are real. This requirement will be satisfied if, when a_r and a_s are conjugate imaginaries, corresponding functions of the pairs $(\bar{u}_r, \bar{v}_r)$, $(\bar{u}_s, \bar{v}_s)$ are given conjugate values; while if a_r is a pure imaginary, the variables $\bar{u}_r$, $\bar{v}_r$ are given conjugate values, as a result of the hypothesis that the functions of (5) are a normal set.

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* Poincaré, *Leçons de Mécanique Céleste*, tome 1, pp. 2, 3; *Les Méthodes Nouvelles*, tome 1, pp. 368, 369.

If a_r, a_s are conjugates, suppose

$$\begin{aligned} a_r &= q + iq' \qquad \bar{u}_r = u' + iu'' \qquad \bar{v}_r = v' + iv'' \\ a_s &= q - iq' \qquad \bar{u}_s = u' - iu'' \qquad \bar{v}_s = v' - iv'' \end{aligned}$$

then from (8)

$$\begin{aligned} du'/dt &= qu' - q'u'' + \phi' \qquad & dv'/dt &= -qv' + q'v'' + \psi' \\ du''/dt &= q'u' + qu'' + \phi'' \qquad & dv''/dt &= -q'v' - qv'' + \psi'' \end{aligned}$$

If a_r is real, and $a_r = q_r$,

$$d\bar{u}_r/dt = q_r\bar{u}_r + \overline{\phi'}_r \qquad d\bar{v}_r/dt = -q_r\bar{v}_r + \overline{\psi'}_r$$

while if a_r is a pure imaginary, $a_r = p\sqrt{-1}$, $p > 0$,

$$\begin{gathered} \bar{u} = x + iy, \quad \bar{v} = x - iy, \\ dx/dt = -py + \overline{\phi'}, \quad dy/dt = px + \overline{\psi'}. \end{gathered}$$

Of the exponents $a_1, a_2, \cdots, a_n$ assume that there are M pure imaginaries, K real, and L pairs of conjugates; then $K + 2L = N$, and $M + N = n$. Equations (8) can then be written in the form

$$\begin{aligned} &dx_i/dt = -p_iy_i + X_i, \quad dy_i/dt = p_ix_i + Y_i \qquad i = 1, \cdots, M, \\ &du_k/dt = q_ku_k + U_k, \qquad dv_k/dt = -q_kv_k + V_k, \qquad k = 1, \cdots, K, \\ (12)\quad &du_r/dt = q_ru_r - q'_ru_{L+r} + U_r, \qquad dv_r/dt = -q_rv_r + q'_rv_{L+r} + V_r \\ &du_{L+r}/dt = q'_ru_r + q_ru_{L+r} + U_{L+r}, \qquad (r = K+1, \cdots, K+L) \\ &\qquad\qquad dv_{L+r}/dt = -q'_rv_r - q_rv_{L+r} + V_{L+r}. \end{aligned}$$

From (8), (9), (10), it is seen that F' has the form

$$(13)\qquad F' = \sum_{i=1}^{M} \pm (x^2{}_i + y^2{}_i) + \sum_{m=1}^{K} u_mv_m + \sum_{r=K}^{K+L} (u_rv_r - u_{L+r}v_{L+r}) + F_3$$

Since the absolute values of the coefficients of (7) are bounded, a number a can be found such that if

$$(D')\quad |x_i| < a, \quad |y_i| < a, \quad |u_r| < a, \quad |v_r| < a, \qquad i = 1, \cdots, M, \ r = 1, \cdots, N,$$

the corresponding variables ξ_i, η_i will satisfy the conditions $|\xi_i| < d'$, $|\eta_i| < d'$.

Since the functions

$$(14)\qquad X_i, \ Y_i, \ U_r, \ V_r, \qquad i = 1, \cdots, M, \quad r = 1, \cdots, N$$

are real, and from (8, 9, 10) are necessarily linear combinations of the first

order partial derivatives of F_3, they possess continuous first order partial derivatives with respect to the variables now considered, which vanish with these variables. If we denote by $g(x_i, y_i, u_r, v_r)$ the largest of the absolute values of the partial derivatives of the functions (14), at a point (x_i, y_i, u_r, v_r) of (D'), then for some τ, $0 \leq \tau \leq 1$,

$$(15) \qquad |X_i| \leq g(\tau x_i, \tau y_i, \tau u_r, \tau v_r) \left[\sum_{i=1}^{M} (|x_i| + |y_i|) + \sum_{r=1}^{N} (|u_r| + |v_r|) \right]$$

and similar inequalities hold for the other functions of (14).

Instead of a single trajectory consider two, represented by the variables

$$x_i, y_i, u_r, v_r; \quad x_i + x'_i, y_i + y'_i, \quad u_r + u'_r, v_r + v'_r;$$

from (12) we obtain

$$\begin{aligned}
&dx'_i/dt = -p_i y'_i + X'_i, \qquad dy'_i/dt = p_i x'_i + Y'_i, \quad i = 1, \cdots, M, \\
&du'_m/dt = q_m u'_m + U'_m, \quad dv'_m/dt = -q_m v'_m + V'_m, \quad m = 1, \cdots, K \\
(16) \quad &du'_r/dt = q_r u'_r - q'_r u'_{L+r} + U'_r, \; dv'_r/dt = -q_r v'_r + q'_r v'_{L+r} + V'_r, \\
&du'_{L+r}/dt = q'_r u'_r + q_r u'_{L+r} + U'_{L+r}, \\
&\qquad dv'_{L+r}/dt = -q'_r v'_r - q_r v'_{L+r} + V'_{L+r}, \\
&\qquad\qquad (r = K+1, \cdots, K+L).
\end{aligned}$$

the primed functions denoting the differences between the corresponding functions (14) for the two points considered. From the equations above we obtain immediately,

$$\begin{aligned}
&\frac{d}{dt}(x'^2_i + y'^2_i) = 2(x'_i X'_i + y'_i Y'_i) \qquad i = 1, \cdots, n. \\
&\frac{du'^2_m}{dt} = 2q_m u'^2_m + 2u'_m U'_m, \\
(17) \qquad &\frac{dv'^2_m}{dt} = -2q_m v'^2_m + 2v'_m V'_m \qquad m = 1, \cdots, K. \\
&\frac{d}{dt}(u'^2_r + u'^2_{L+r}) = 2q_r(u'^2_r + u'^2_{L+r}) + 2(u'_r U'_r + u'_{L+r} U'_{L+r}), \\
&\frac{d}{dt}(v'^2_r + v'^2_{L+r}) = -2q_r(v'^2_r + v'^2_{L+r}) + 2(v'_r V'_r + v'_{L+r} V'_{L+r}) \\
&\qquad\qquad (r = K+1, \cdots, K+L).
\end{aligned}$$

As before,

$$(18) \quad |X'_i| \leq g(x_i + \tau x'_i, y_i + \tau y'_i, u_r + \tau u'_r, v_r + \tau v'_r) \left[\sum_{i=1}^{M} (|x'_i| + |y'_i|) + \sum_{r=1}^{N} (|u'_r| + |v'_r|) \right] \qquad 0 < \tau < 1.$$

with similar inequalities for the other functions.

Part II. *Extension of the theorems of Bohl.*

1. Some preliminary inequalities.

The paragraphs which follow are very similar to the corresponding ones in the memoir of Bohl.* Suppose $l_1, l_2, \cdots, l_M, m_1, m_2, \cdots, m_N, m_{K+L+r} = m_{K+r}$ any positive numbers, and $s_1, s_2, \cdots, s_M, t_1, t_2, \cdots, t_M, z_1, z_2, \cdots, z_N$ a set of real variables. Then from the inequality

$$\sqrt{m}\sqrt{\sum_{i=1}^{m} x^2_i} \geq \sum_{i=1}^{m} |x_i|$$

and the introduction of the constants

$$\bar{k} \leq l_i, m_r, \quad \bar{g} \geq l_i, m_r, \quad \Gamma = \frac{1}{\bar{k}}\sqrt{2M+N}, \quad \Delta = \bar{g}\sqrt{2M+N}$$

we obtain

$$\Gamma\sqrt{\sum_{i=1}^{M} l^2_i(s^2_i+t^2_i) + \sum_{r=1}^{N} m^2_r z^2_r} \geq \sum_{i=1}^{M}(|s_i|+|t_i|) + \sum_{r=1}^{N}|z_r| \tag{19}$$

$$\Delta\sqrt{\sum_{i=1}^{M} l^2_i(s^2_i+t^2_i) + \sum_{r=1}^{N} m^2_r z^2_r} \geq \sum_{i=1}^{M} l^2_i(|s_i|+|t_i|) + \sum_{r=1}^{N} m^2_r|z_r|$$

By choosing the constant a defining (D') sufficiently small we can make the quantity $g(x_i, y_i, u_r, v_r)$ arbitrarily small; hence if λ is a positive quantity to be restricted later, we may assume $g < \lambda$. For convenience suppose $q_{K+r} = q_{K+L+r}$. If

$$r' = \sum_{i=1}^{M} l^2_i(x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r - \sum_{r=1}^{N} u'^2_r$$

we obtain from (17),

$$\frac{1}{2}\frac{dr'}{dt} = -\sum_{r=1}^{N} m^2_r q_r v'^2_r - \sum_{r=1}^{N} q_r u'^2_r + \sum_{i=1}^{M} l^2_i(x'_i X'_i + y'_i Y'_i) + \sum_{r=1}^{N} m^2_r v'_r V'_r - \sum_{r=1}^{N} u'_r U'_r.$$

Since

$$|X'_i|, |Y'|, |U'_r|, |V'_r| \leq \lambda\left[\sum_{i=1}^{M}(|x'_i|+|y'_i|) + \sum_{r=1}^{N}(|u_r'|+|v'_r|)\right]$$

and

$$\sum_{i=1}^{M}(|x'_i|+|y'_i|) + \sum_{r=1}^{N}|v'_r| \leq \Gamma\sqrt{\sum_{i=1}^{M} l^2_i(x'^2_i+y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r}$$

$$\sum_{r=1}^{N}|u'_r| \leq \sqrt{N}\sqrt{\sum_{r=1}^{N} u'^2_r}$$

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* *L. c.*, II, § 6.

we obtain

$$|X'_i|, |Y'_i|, |U'_r|, |V'_r| \leq \lambda \left[\Gamma\sqrt{\sum_{i=1}^{M} l^2_i(x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r} + \sqrt{N}\sqrt{\sum_{r=1}^{N} u'^2_r}\right]$$

Consequently if $r' \leq 0$ when $t = t_0$,

$$\frac{1}{2}\frac{dr'}{dt} \leq -\sum_{r=1}^{N} q_r u'^2_r + \lambda(\Gamma + \sqrt{N})\sqrt{\sum_{r=1}^{N} u'^2_r} \left[\sum_{i=1}^{M} l^2_i(|x'_i| + |y'_i|) + \sum_{r=1}^{N}(m^2_r|v'_r| + |u'_r|)\right]$$

Since

$$\sum_{i=1}^{M} l^2_i(|x'_i| + |y'_i|) + \sum_{r=1}^{N} m^2_r |v'_r| \leq \Delta\sqrt{\sum_{i=1}^{M} l^2_i(x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r} \leq \Delta\sqrt{\sum_{r=1}^{N} u'^2_r}$$

we have

$$\frac{1}{2}\frac{dr'}{dt} \leq -\sum_{r=1}^{N} q_r u'^2_r + \lambda(\Gamma + \sqrt{N})(\Delta + \sqrt{N})\sum_{r=1}^{N} u'^2_r.$$

Then if q is the smallest of the quantities $q_1, q_2, \cdots, q_N$,

$$\frac{1}{2}\frac{dr'}{dt} \leq -[q - \lambda(\Gamma + \sqrt{N})(\Delta + \sqrt{N}]\sum_{r=1}^{N} u'^2_r.$$

or if $\lambda(\Gamma + \sqrt{N})(\Delta + \sqrt{N}) < q$, then $dr'/dt \leq 0$.

Suppose this inequality for λ satisfied within (D'). Then if the two trajectories lie in (D') for $t_0 \leq t \leq t_1$, and $r' \leq 0$ for $t = t_0$, then $dr'/dt < 0$ also if $\sum_{i=1}^{N} u'^2_r \neq 0$. Suppose $r' \geq 0$ for $t = t_1$,; then a time $t' \leq t_1$ must exist such that $r' = 0$, $t = t'$, and $r' < 0$, $t_0 < t < t'$. Consequently there exists also a time $t'' < t'$ at which $dr'/dt = 0$; but at this time $r' < 0$, hence $dr'/dt < 0$. From this contradiction we obtain the result that $r' < 0$ when $t = t_1$, and $r' < 0$, $t_0 < t \leq t_1$.

Consequently from the definition of r', $\sum_{r=1}^{N} u'^2_r > 0$, $t_0 < t \leq t_1$, and if $\sum_{r=1}^{N} u'^2_r \neq 0$ for $t = t_0$, then $\sum_{r=1}^{N} u'^2_r > 0$, $t_0 \leq t \leq t_1$.

Under these assumptions compute $d/dt \sum_{r=1}^{N} u'^2_r$:

$$\frac{1}{2}\frac{d}{dt}\sum_{r=1}^{N} u'^2_r = \sum_{r=1}^{K}(q_r u'^2_r + u'_r U'_r)$$

$$+ \sum_{r=K+1}^{K+L}[q_r(u'^2_r + u'^2_{L+r}) + u'_r U'_r + u'_{L+r}U'_{L+r}]$$

$$= \sum_{r=1}^{N} q_r u'^2_r + \sum_{r=1}^{N} u'_r U'_r$$

Since

$$\left|\sum_{r=1}^{N} u'_r U'_r\right| \leq \left(\sum_{r=1}^{N}|u'_r|\right)\lambda\left[\sum_{i=1}^{M}(|x'_i|+|y'_i|) + \sum_{r=1}^{N}(|u'_r|+|v'_r|)\right]$$

$$\leq \lambda\sqrt{N}\sqrt{\sum_{r=1}^{N} u'^2_r}\ (\Gamma+\sqrt{N})\sqrt{\sum_{r=1}^{N} u'^2_r}$$

$$\leq \lambda\sqrt{N}\,(\Gamma+\sqrt{N})\sum_{r=1}^{N} u'^2_r$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{r=1}^{N} u'^2_r \geq [q-\lambda\sqrt{N}(\Gamma+\sqrt{N})]\sum_{r=1}^{N} u'^2_r$$

Suppose $\sigma > 2\lambda\sqrt{N}(\Gamma+\sqrt{N})$, while $\sigma < 2q$; then

$$\frac{d}{dt}\sum_{r=1}^{N} u'^2_r > (2q-\sigma)\sum_{r=1}^{N} u'^2_r$$

and

$$\left[\sum_{r=1}^{N} u'^2_r\right]_{t=t_1} > e^{(2q-\sigma)(t_1-t_0)}\left[\sum_{r=1}^{N} u'^2_r\right]_{t=t_0}.$$

Consequently we have the following theorem: Given a set of positive numbers σ, $l_1, l_2, \cdots, l_M$, $m_1, m_2, \cdots, m_N$, with $m_{K+L+r} = m_{K+r}$, then if a neighborhood (D') of the origin is so chosen that within it the partial derivatives of X_i, Y_i, U_r, V_r are less in absolute value than either of the quantities

$$\frac{q}{(\Gamma+\sqrt{N})(\Delta+\sqrt{N})} \quad , \quad \frac{\sigma}{2\sqrt{N}(\Gamma+\sqrt{N})}$$

where Γ, Δ are the quantities previously defined, this neighborhood has the following properties: 1. If the coördinates of two trajectories lie in (D') for $t_0 \leq t \leq t_1$ and if when $t = t_0$

$$r' = \sum_{i=1}^{M} l^2_i(x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r v'^2_r - \sum_{r=1}^{N} u'^2_r \leq 0, \quad \sum_{r=1}^{N} u'^2_r \neq 0,$$

then $r' < 0$ for the interval $t_0 < t \leq t_1$; and

$$\left[\sum_{r=1}^{N} u'^2_r\right]_{t=t_1} > e^{(2q-\sigma)(t_1-t_0)} \left[\sum_{r=1}^{N} u'^2_r\right]_{t=t_0}$$

2. If the coördinates defining two trajectories lie in (D') for $t_0 \geq t \geq t_1$ $(t_1 < t_0)$ and if for $t = t_0$,

$$s' = \sum_{i=1}^{M} l^2_i (x'^2_i + y'^2_i) + \sum_{r=1}^{N} m^2_r u'^2_r - \sum_{r=1}^{N} v'^2_r \leq 0, \quad \sum_{r=1}^{N} v'^2_r \neq 0,$$

then $s' < 0$ for the interval $t_0 > t \geq t_1$, and

$$\left[\sum_{r=1}^{N} v'^2_r\right]_{t=t_1} > e^{(2q-\sigma)(t_0-t_1)} \left[\sum_{r=1}^{N} v'^2_r\right]_{t=t_0}$$

The last part of this theorem can be demonstrated in the same manner as the first.*

2. Certain inequalities obtained from the energy integral.

In this and the following paragraphs it will be assumed that the functions ϕ_i, ψ_i in (2) are constants, and that the function F does not contain the time, consequently the integral $F = C$ exists; it will also be assumed that the quadratic form $\sum_{i=1}^{M} \pm (x^2_i + y^2_i)$ in (13) is definite. Then if $\mu > 0$ is given, the constant a defining (D') can be chosen so small that within this region, (E),

$$| F_3 | \leq \mu \left[\sum_{i=1}^{M} (x^2_i + y^2_i) + \sum_{r=1}^{N} (u^2_r + v^2_r)\right] .$$

Suppose δ a positive number such that the region defined by the inequalities

$$(F_1) \qquad \sum_{i=1}^{M} (x^2_i + y^2_i) \leq \delta^2$$

$$(G) \qquad \sum_{r=1}^{N} u^2_r \leq \delta^2/4, \quad \sum_{r=1}^{N} v^2_r \leq \delta^2/4$$

* While it is almost self-evident that no coördinate can become indeterminate for a value $\bar{t}$ on the interval $t_0 < t < t_1$ under these hypotheses, a proof can be given as follows. Suppose M' an upper bound for the right-hand members of (12), and $|x_i|$, $|y_i|$, $|u_r|$, $|v_r| < b$ in the neighborhood considered. Then if t' is any value of t, the method of successive approximations of Picard defines these coördinates within an interval $|t'' - t'| < b/M'$. Since any finite value of $t - t_0$ is less than some integral multiple of b/M', the values of the coördinates for this time $\bar{t}$ are defined by the method of Picard, and cannot therefore be indeterminate when $t \to \bar{t}$, if it is assumed that the cordinates lie in the given region for $t_0 \leq t \leq \bar{t} + \bar{h}$, $\bar{h} > 0$.

lies within (E). Introduce also the region

$$(F_2) \qquad \sum_{r=1}^{M} (x^2_i + y^2_i) \leq 2\delta^2/9.$$

Then from the energy integral and (13) we obtain the theorem: If the trajectory (x_i, y_i, u_r, v_r) lies in the region (F_1, G) for $t = t_1$, and in (F_2, G) for $t = t_2$, then if $\mu < \frac{1}{8}$,

$$\left[\sum_{i=1}^{M} (x^2_i + y^2_i) \right]_{t=t_1} < \delta^2.$$

For from (13),

$$\left[\sum_{i=1}^{M} (x^2_i + y^2_i) \right]_{t=t_1}$$
$$= \pm \left[\pm \sum_{i=1}^{M} (x^2_i + y^2_i) + \sum_{m=1}^{K} u_m v_m + \sum_{r=K+1}^{K+L} (u_r v_r - u_{L+r} v_{L+r}) + F_3 \right]_{t=t_2}$$
$$\mp \left[\sum_{m=1}^{K} u_m v_m + \sum_{r=K+1}^{K+L} (u_r v_r - u_{L+r} v_{L+r}) + F_3 \right]_{t=t_1}$$

Hence from the inequalities $u_k v_k \leqq \frac{1}{2}(u^2_k + v^2_k)$, $-u_r v_r \leq \frac{1}{2}(u^2_r + v^2_r)$, we obtain

$$\left[\sum_{i=1}^{M} (x^2_i + y^2_i) \right]_{t=t_1} \leq \frac{2\delta^2}{9} + \frac{\delta^2}{2} + \mu \left[2\delta^2 + \frac{2\delta^2}{9} \right]$$
$$\leq \delta^2 [\tfrac{2}{9} + \tfrac{1}{2} + \mu(2 + \tfrac{2}{9})]$$

and if $\mu < \frac{1}{8}$, the theorem follows.

3. Introduction of certain domains and inequalities.

Suppose (H) a region lying within (E) and such that in (H) the partial derivatives of the functions (14) are less in absolute value than θq, where

$$\theta < \frac{1}{2\sqrt{N}(\sqrt{2M} + \sqrt{N})}$$

Assume d so small that the region

$$\sum_{i=1}^{M} (x^2_i + y^2_i) \leq d^2, \quad \sum_{r=1}^{N} u^2_r \leq d^2/4, \quad \sum_{r=1}^{N} v^2_r \leq d^2/4$$

lies within (H), and construct the regions

$$(A) \qquad \sum_{i=1}^{M} (x^2_i + y^2_i) \leq d^2,$$

$$(a) \qquad \sum_{i=1}^{M} (x^2_i + y^2_i) \leq 2d^2/9,$$

$$(B) \qquad \sum_{r=1}^{N} u^2_r \leq d^2/4, \quad \sum_{r=1}^{N} v^2_r \leq d^2/4.$$

Then we have the theorem: If (x_i, y_i, u_r, v_r) lies in (a, B) for $t = t_2$, in (A, B) for $t = t_1$, and if for $t = t_1$, $\sum_{r=1}^{N} u^2_r = d^2/4$, then $d/dt \sum_{r=1}^{N} u^2_r > 0$.

For, from the preceding paragraph, if $t = t_1$

$$\sum_{i=1}^{M} (x^2_i + y^2_i) < d^2.$$

From (12) or (16) we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{r=1}^{N} u^2_r = \sum_{r=1}^{K} (q_r u^2_r + u_r U_r) + \sum_{r=K+1}^{K+L} [q_r(u^2_r + u^2_{L+r}) + u_r U_r + u_{L+r} U_{L+r}]$$

$$= \sum_{r=1}^{N} q_r u^2_r + \sum_{r=1}^{N} u_r U_r.$$

From the inequalities

$$\left|\sum_{r=1}^{N} u_r U_r\right| \leq \left(\sum_{r=1}^{N} |u_r|\right) \theta q \left[\sum_{i=1}^{M} (|x_i| + |y_i|) + \sum_{r=1}^{N} (|u_r| + |v_r|)\right]$$

$$\sum_{i=1}^{M} (|x_i| + |y_i|) \leq \sqrt{2M} \sqrt{\sum_{i=1}^{M} (x^2_i + y^2_i)}$$

$$\sum_{r=1}^{N} (|u_r| + |v_r|) \leq \sqrt{2N} \sqrt{\sum_{r=1}^{N} u^2_r + v^2_r)}$$

$$\sum_{r=1}^{N} |u_r| \leq \sqrt{N} \sqrt{\sum_{r=1}^{N} u^2_r}$$

we obtain

$$\left|\sum_{r=1}^{N} u_r U_r\right| \leq \theta q \sqrt{N} \sqrt{\sum_{r=1}^{N} u^2_r} \left[\sqrt{2M} \sqrt{\sum_{i=1}^{M} (x^2_i + y^2_i)} + \sqrt{2N} \sqrt{\sum_{r=1}^{N} (u^2_r + v^2_r)}\right]$$

$$\leq \theta q \sqrt{N} \frac{d}{2} [\sqrt{2M}\, d + \sqrt{2N} \sqrt{d^2/2}]$$

$$\leq \theta q d^2 \frac{\sqrt{N}}{2} [\sqrt{2M} + \sqrt{N}] < q \frac{d^2}{4}$$

Also,

$$\sum_{r=1}^{N} q_r u^2_r \geq q \sum_{r=1}^{N} u^2 = q \frac{d^2}{4}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\sum_{r=1}^{N} u^2_r > q\frac{d^2}{4} - q\frac{d^2}{4} = 0.$$

In the same manner can be proved the theorem: If (x_i, y_i, u_r, v_r) lies in (a, B) for $t = t_1$, in (A, B) for $t = t_2$, then if for $t = t_1$, $\sum_{r=1}^{N} v^2_r = d^2/4$,

$$\frac{d}{dt}\sum_{r=1}^{N} v^2_r < 0, \quad t = t_1.$$

4. Existence theorems.

For convenience the domains introduced above can be grouped in the form

$$(A, B) \qquad \sum_{i=1}^{M}(x^2_i + y^2_i) \leq d^2, \quad \sum_{r=1}^{N} u^2_r \leq d^2/4, \quad \sum_{r=1}^{N} v^2_r \leq d^2/4,$$

$$(a, B) \qquad \sum_{i=1}^{M}(x^2_i + y^2_i) \leq 2d^2/9, \quad \sum_{r=1}^{N} u^2_r \leq d^2/4, \quad \sum_{r=1}^{N} v_r{}^2 \leq d^2/4.$$

We shall prove the following theorem: To every set of values of the region (R),

$$\sum_{i=1}^{M}(x^2_i + y^2_i) \leq 2d^2/9, \quad \sum_{r=1}^{N} v^2_r \leq d^2/4$$

can be adjoined a set of u-values of the region $\sum_{r=1}^{N} u^2_r \leq d^2/4$ such that the set of values (x_i, y_i, v_r), (u_r) are the initial values of a trajectory which remains in the region (A, B) for $t > t_0$.

The theorem will be true if it can be shown that the contrary assumption leads to a contradiction. Suppose $(x_i, y_i, v_r)_0$ a set of values of (R), such that however a set of values $(u)_0$ is chosen satisfying $\sum_{r=1}^{N} u^2_r \leq d^2/4$, the trajectory possessing the initial values $(x_i, y_i, v_r)_0$, $(u)_0$ ceases to satisfy the inequalities of (A, B) for $t > \tau$; in other words, inequalities (A, B) are satisfied for $t_0 \leq t \leq \tau$, while however small h may be, these inequalities are not satisfied in the entire interval $t_0 \leq t < \tau + h$. Consider the values of $u_1, u_2, \cdots, u_N$ when $t = \tau$; if this set is denoted by (u), it will be shown that (u) lies on the boundary of the region $\sum_{r=1}^{N} u^2_r \leq d^2/4$.

The set of values (x_i, y_i, u_r, v_r) lies on the boundary of (A, B) for $t = \tau$ and since the trajectory lies in (a, B) for $t = t_0$, we have from the results of a preceding paragraph $\sum_{i=1}^{M}(x^2_i + y^2_i) < d^2$. Also, if the equation

$\sum_{r=1}^{N} v^2_r = d^2/4$ is satisfied, then $d/dt \sum_{r=1}^{N} v^2_r < 0$; and if the set of values (u) did not lie on the boundary, $(\sum_{r=1}^{N} u^2_r < d^2/4)$, it would follow that the trajectory would remain in the region (A, B) for some interval of time, contrary to hypothesis. Hence to each set $(u)_0$ corresponds a set (u) on the boundary of the region $\sum_{r=1}^{N} u^2_r \leq d^2/4$. It will be shown that this correspondence is a continuous one.*

Suppose first that the set $(u)_0$ lies in the interior of the region, and consequently $\sum_{r=1}^{N} u^2_r < d^2/4$; then $\tau > t_0$. Then a constant h can be determined such that the trajectory with initial values $(x_i, y_i, v_r)_0 (u)_0$ lies within (H) for $t_0 \leq t \leq \tau + h$, $\sum_{r=1}^{N} u_r{}^2 > d^2/4$ for $t = \tau + h$, and $t_0 < \tau - h$. The number h can be chosen arbitrarily small. In the interval $t_0 \leq t \leq \tau - h$ the inequality $\sum_{r=1}^{N} u^2_r < d^2/4$ is constantly satisfied, otherwise the trajectory would not satisfy $\sum_{r=1}^{N} u^2_r \leq d^2/4$ for $t_0 \leq t \leq \tau$, from the inequality $d/dt \sum_{r=1}^{N} u^2_r > 0$ on the boundary of this region.

In the interval $t_0 \leq t \leq \tau - h$, then, $d^2/4 - \sum_{r=1}^{N} u^2_r \geq \beta^2 > 0$; consequently a region (ω) lying in the interior of the region $\sum_{r=1}^{N} u^2_r \leq d^2/4$ can be found such that a trajectory with the initial values $(x_i, y_i, v_r)_0$ and any set of values of (ω) lies in (H) for $t_0 \leq t \leq \tau + h$; for $t = \tau + h$ its u-values satisfy $\sum_{r=1}^{N} u^2_r > d^2/4$, while for $t = \tau - h$ the inequality $\sum_{r=1}^{N} u^2_r < d^2/4$ is satisfied.

Hence to each set of values of (ω) corresponds a value $\bar{\tau}$, and $\bar{\tau} > \tau - h$ since $\sum_{r=1}^{N} u^2_r < d^2/4$, $t_0 \leq t \leq \tau - h$. Also, $\bar{\tau} < \tau + h$ since for $t = \tau + h$, $\sum_{r=1}^{N} u^2_r > d^2/4$. The set of values (u) corresponding to a set belonging to (ω) is formed for $t = \bar{\tau}$. Since h is arbitrarily small, it follows that by

* The following argument is based on the continuity of the solutions of a differential system with respect to the initial values of the variables; see, for example, Goursat, *Cours d'Analyse*, tome III, Chap. 23, (1915).

choosing the neighborhood (ω) sufficiently small the u-coördinates of a trajectory with initial values $(x_i, y_i, v_r)_0$, (u) in (ω) can be made to differ as little as may be desired from their values on the trajectory with initial values $(x_i, y_i, v_r)_0$, $(u)_0$ at the same time; and the u-coördinates of this trajectory in the interval $\tau - h \leq t \leq \tau + h$ differ as little as may be desired from their values for $t = \tau$. Hence a neighborhood of the set $(u)_0$ can be found such that the set (u') which corresponds to a point $(u')_0$ of this neighborhood and the set (u'') which corresponds to a point $(u'')_0$ of the same neighborhood will satisfy the condition $| u''_r - u'_r | < \epsilon$ $(r = 1, \cdots, N)$, where ϵ has been chosen arbitrarily small. The continuity of this correspondence for points $(u)_0$ in the interior of the region $\sum_{r=1}^{N} u^2_r \leqq d^2/4$ follows; the proof for points on the boundary can be given in the same manner.

Consequently to each point in the region $\sum_{r=1}^{N} u^2_r \leq d^2/4$ corresponds a point $U_1, U_2, \cdots, U_N$ on the boundary, while points on the boundary correspond to themselves,, and the coördinates U_r are continuous functions of the variables $u_1, u_2, \cdots, u_N$. But such a correspondence is impossible, as a consequence of a theorem of Bohl.*. Hence the theorem.

In the same manner can be proved the theorem: To every set of values (x_i, y_i, u_r) of the region

$$\sum_{i=1}^{M} (x^2_i + y^2_i) \leq 2d^2/9, \quad \sum_{r=1}^{N} u^2_r \leq d^2/4$$

corresponds a set (v) of the region $\sum_{r=1}^{N} v^2_r \leq d^2/4$ such that the trajectory with this set of initial values at $t = t_0$ lies in the region (A, B) for $t < t_0$.

5. On the manifold of the trajectories whose existence has been established.

Suppose a neighborhood (H') determined such that (H') has the properties of both (D') and (H); such a neighborhood can be determined, with $l_1 = l_2 = \cdots = l_M = 1$, and $m_1 = m_2 = \cdots = m_N = m < 1$. Then suppose two trajectories to lie within (H') for $t \geq t_0$; then if in the terminology of § 1, at $t' \geq t_0$.

$$\sum_{i=1}^{M} (x'^2_i + y'^2_i) + m^2 \sum_{r=1}^{N} v'^2_r - \sum_{r=1}^{N} u'^2_r < 0,$$

* *L. c.*, p. 198. On the boundary the functions U_r cannot vanish simultaneously.

from the theorem of § 1 it would follow that $\sum_{r=1}^{N} u'^2_r$ increases indefinitely with t. Consequently for $t \geq t_0$,

$$\sum_{i=1}^{M} (x'^2_i + y'^2_i) + m^2 \sum_{r=1}^{N} v'^2_r \geqq \sum_{r=1}^{N} u'^2_r.$$

Hence the theorem: If two trajectories lie in (H') for $t \geq t_0$, and if for some $t' \geq t_0$, the coördinates (x_i, y_i, v_r) coincide, the two trajectories coincide.

If the u-coördinates corresponding to a given set of values of x_i, y_i, v_r considered as functions of these coördinates, are denoted by $[u_1], [u_2], \cdots, [u_N]$, the inequality above proves the continuity of these functions. In the same manner can be obtained the theorem: If two trajectories lie in (H') for $t < t_0$ and if for some $t'' \leq t_0$ the x_i, y_i, u_r coördinates coincide, the two trajectories coincide.

The v-coördinates of a trajectory which lies within (H') for $t < t_0$ will be denoted by $[v_1], [v_2], \cdots, [v_N]$; the continuity of these functions follows as before.

6. Trajectories which lie in (H') for all values of t.

Suppose fixed a set of values (x_i, y_i) satisfying (a), (§ 3), and construct the functions

$$u_r - [u_r], \quad v_r - [v_r] \quad r = 1, \cdots, N.$$

These are continuous functions of the variables u_r, v_r in the region

$$(B) \qquad \sum_{r=1}^{N} u^2_r \leq d^2/4, \quad \sum_{r=1}^{N} v_r^2 \leq d^2/4.$$

and consequently from a theorem of Bohl,* there exists a point on the boundary of (B) for which, unless these functions vanish simultaneously,

$$u_r - [u_r] = -\nu u_r, \quad v_r - [v_r] = -\nu v_r, \quad \nu > 0, \quad r = 1, \cdots, N,$$

consequently

$$\sum_{r=1}^{N} u^2_r = d^2/4 \text{ or } \sum_{r=1}^{N} v^2_r = d^2/4.$$

From

$$(1+\nu)u_r = [u_r], \quad (1+\nu)v_r = [v_r] \quad r = 1, \cdots, N$$

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* *L. c.*, pp. 199, 200. The theorem of Bohl is proved for a slightly different domain, but the proof is sufficiently general to cover the case considered here, as is immediately evident; instead of his region (G) we have (B), and a corresponding interpretation of his functions $\phi_i (i = 1, \cdots, n)$.

it follows that

$$(1+\nu)^2 \sum_{r=1}^{N} u^2{}_r = \sum_{r=1}^{N} [u_r]^2, \quad (1+\nu)^2 \sum_{r=1}^{N} v^2{}_r = \sum_{r=1}^{N} [v_r]^2$$

and therefore

$$(1+\nu)^2\, d^2/4 = \sum_{r=1}^{N} [u_r]^2 \quad \text{or} \quad (1+\nu)^2\, d^2/4 = \sum_{r=1}^{N} [v_r]^2$$

Since $\sum_{r=1}^{N} [u_r]^2 \leq d^2/4$, $\sum_{r=1}^{N} [v_r]^2 \leq d^2/4$, we have arrived at a contradiction unless the existence is admitted of a set of values for which

$$u_r = [u_r], \quad v_r = [v_r], \quad r = 1, \cdots, N.$$

Consequently we have the theorem: To every set of values of the region

$$\sum_{r=1}^{M} (x^2{}_i + y^2{}_i) \leq 2d^2/9$$

corresponds a set of the region

$$\sum_{r=1}^{N} u^2{}_r \leq d^2/4, \quad \sum_{r=1}^{N} v^2{}_r \leq d^2/4$$

such that the trajectory with these initial values (x_i, y_i), (u_r, v_r) at $t = t_0$ remains in the region (A, B) for all values of the time.

Consider two trajectories which remain in (A, B) for all values of t. From § 3 we obtain the inequalities

$$\sum_{i=1}^{M} (x'^2{}_i + y'^2{}_i) + m^2 \sum_{r=1}^{N} v'^2{}_r \geq \sum_{r=1}^{N} u'^2{}_r$$

$$\sum_{i=1}^{M} (x'^2{}_i + y'^2{}_i) + m^2 \sum_{r=1}^{N} u'^2{}_r \geq \sum_{r=1}^{N} v'^2{}_r \qquad m < 1,$$

Therefore

$$\sum_{i=1}^{M} (x'^2{}_i + y'^2{}_i) \geq \frac{1-m^2}{2} \sum_{r=1}^{N} (u'^2{}_r + v'^2{}_r).$$

Hence if for any value of t, the (x_i, y_i) coördinates coincide, the two trajectories coincide.

Expansion Problems in Connection with the Hypergeometric Differential Equation.*

By Bernhard Paul Reinsch.

I. Introduction.

1. *Statement of the Problem.* The object of this paper is the development of an expansion theory for the generalized hypergeometric functions of Goursat † and Pochhammer,‡ and for the ordinary hypergeometric functions § of Gauss in particular, analogous to the one worked out by Neumann and Gegenbauer for Bessel functions.‖

We first treat the expansion of $f(x)$, an arbitrary analytic function of the complex variable x, in series of ordinary hypergeometric functions,

$$\bar{F}_{k+m}(x) = x^{k+m}F(a+k+m,\ b,\ 1+k+m\,;\ x), \quad (m = 0,\ 1,\ 2,\ \cdots),$$

which are particular solutions of

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0,$$

where $\alpha = a$, $\beta = b-k-m$, and $\gamma = 1-k-m$. The quantities a, b, and k are to be any fixed numbers, real or complex, subject to the sole restriction that k shall not be a negative integer.

This expansion is at once developed in a generalized form, analogous to Gegenbauer's generalization of Neumann's expansion in terms of Bessel functions, since this includes as a special case (i. e. $k = 0$) the analogue of Neumann's expansion. Some important properties of $\bar{F}_{k+m}(x)$ and of $G_{m,k}(t)$, the polynomials associated with $\bar{F}_{k+m}(x)$, are then discussed.

This expansion theory is then extended to the generalized hypergeometric functions in Section III, and to the expansion of functions of several variables in Section IV.

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† *Ann. de l'École Normale*, Ser. 2, Vol. 12 (1883), p. 261 and p. 395.

‡ *Mathematische Annalen*, Vol. 38 (1891), p. 586.

§ Darboux, in "Sur l'approximation des fonctions de très-grands nombres, etc.", *Journal de Mathématiques*, Ser. 3, Vol. 4 (1878), p. 377, treated a special case. He developed the expansions of analytic functions in terms of Jacobi polynomials, $F(-n,\ \alpha+n,\ \gamma;\ x)$, by the method of Neumann.

‖ See Watson, *Theory of Bessel Functions* (London, 1922), Chapters 9 and 16, where an excellent summary of these results is given. Here will also be found additional references.

II. The Expansion of an Arbitrary Analytic Function in Terms of Solutions of the Ordinary Hypergeometric Differential Equation of Gauss.

2. *Expansion of x^{k+m} in a Series of $\bar{F}_{k+m}(x)$.* To determine the coefficients c_m in the expansion

$$x^{k+m} = \sum_{m=0}^{\infty} c_m \bar{F}_{k+m}(x),$$

we insert in the right member the expansion of $\bar{F}_{k+m}(x)$,

$$\bar{F}_{k+m}(x) = x^{k+m} + \frac{(a+k+m)b}{1.(1+k+m)}\, x^{k+m+1}$$
$$+ \frac{(a+k+m)(a+k+m+1)b(b+1)}{1.2(1+k+m)(2+k+m)}\, x^{k+m+2} + \cdots, \quad (m = 0, 1, 2, \cdots),$$

and then equate coefficients of like powers of x. The resulting expansion,

$$(1) \qquad x^{k+m} = \sum_{s=0}^{\infty} \frac{(a+k+m)_s (-b)_s}{s!\,(1+k+m)_s}\, \bar{F}_{k+m+s}(x),$$

where $(a)_s = a(a+1)(a+2)\cdots(a+s-1)$ and $(a)_0 = 1$, we shall show to be absolutely and uniformly convergent and to represent the function x^{k+m}.

3. *An Expression Dominating $\bar{F}_{k+m}(x) = x^{k+m}F(a+k+m,\ b,\ 1+k+m;\ x)$.* Always assuming k not to be a negative integer, we have

$$|\bar{F}_{k+m}(x)| = |x|^{k+m}\,|F_{k+m}(x)|$$
$$\leqq |x|^{k+m}\left[1 + \frac{|a+k+m|.|b|}{|1+k+m|.\ 1}\,|x|\right.$$
$$\left. + \frac{|a+k+m|.|a+k+m+1|.|b|.|b+1|}{|1+k+m|.|1+k+m+1|.\ 2!}\,|x|^2 + \cdots\right],$$

where the series in the brackets converges absolutely and uniformly for all finite values of m and for all values of x such that $|x| \leqq R' < 1$. We shall show that $|F_{k+m}(x)|$ possesses an upper bound M for *all* (positive integral) values of m and for all values of x such that $|x| \leqq R' < 1$. It is possible to choose m large enough $(m = m_2 \geqq m_1)$ so that $R(a+k+m_2) > 0$, $R(1+k+m_2) > 0$, and so that for *every* m_2 equal to or greater than m_1, only one of the following two cases exists. *Either*

$$\frac{|a+k+m_1+\nu|}{|1+k+m_1+\nu|} \leqq \frac{|a+k+m_2+\nu|}{|1+k+m_2+\nu|} \leqq 1, \quad (\nu = 0, 1, 2, \cdots),$$

or

$$1 \leqq \frac{|a+k+m_2+\nu|}{|1+k+m_2+\nu|} \leqq \frac{|a+k+m_1+\nu|}{|1+k+m_1+\nu|}, \qquad (\nu = 0, 1, 2, \cdots).$$

In the *first case,* we have

$$|F_{k+m_2}(x)| \leqq [1 + \frac{|b|}{1}|x| + \frac{|b|.(|b|+1)}{2!}|x|^2 + \cdots]$$
$$= [1-|x|]^{-|b|}, \qquad (m_2 \geqq m_1),$$

which certainly has an upper bound M_1, for all values of m_2 and for all values of x such that $|x| \leqq R' < 1$.

In considering the *second case,* we note that the expansion of $F_{k+m_1}(x)$ converges absolutely and uniformly, and hence $|F_{k+m_1}(x)|$ has an upper bound M_2 when $|x| \leqq R' < 1$, where m_1 is a value of m such that $|a+k+m+\nu| \geqq |1+k+m+\nu|$, $(\nu = 0, 1, 2, \cdots)$. But since

$$(2) \quad |F_{k+m_1}(x)| \leqq [1 + \frac{|a+k+m_1|.|b|}{|1+k+m_1|.1}|x|$$
$$+ \frac{|a+k+m_1|.|a+k+m_1+1|.|b|.|b+1|}{|1+k+m_1|.|1+k+m_1+1|.2!}|x|^2 + \cdots] \leqq M_2,$$

and since the replacing of m_1 by m_2, where $m_2 > m_1$, in the series (2) gives a series whose terms are less than or equal to the corresponding terms in (2), it follows that for all values of m greater than m_1, we have

$$|F_{k+m}(x)| \leqq M_2, \qquad (m > m_1).$$

It is evident that for each value of m less than m_1, $|F_{k+m}(x)|$ has an upper bound in the region $|x| \leqq R' < 1$. In this group of upper bounds let M_3 be the largest. Then, letting M be the greatest of the quantities M_1, M_2, and M_3, we have

$$(3) \qquad |F_{k+m}(x)| \leqq |x|^{k+m} M$$

for *all* values of m and for all values of x such that $|x| \leqq R' < 1$.

4. *Convergence of the Expansion for* x^{k+m}. By means of the last inequality, it may be seen that the series in the second member of (1) is dominated by the series

$$(4) \qquad \sum_{s=0}^{\infty} \left| \frac{(a+k+m)_s(-b)_s}{s!\,(1+k+m)_s} \right| .|x|^{k+m+s} M.$$

Since the ratio of the $(s+1)$th term to the sth term approaches $|x|$ as s approaches infinity, we see that (4), and hence (1), is absolutely convergent when $|x| \leqq R' < 1$. After dividing out the factor x^k, we can prove the

uniform convergence of the series in (1) throughout the region $|x| \leqq R' < 1$, for then (4) divided by x^k will be dominated by a convergent series of positive constants independent of x. Hence, the uniform convergence follows from the test of Weierstrass.

Then, since the power series expansions of $\bar{F}_{k+m}(x)$ all converge absolutely in the circle $|x| \leqq R' < 1$, and (1) divided by x^k converges uniformly in the same circle, we know from the Weierstrass theorem on double series that the rearrangement in deriving (1) was valid and that x^m, and consequently x^{k+m}, is represented by its expansion in the circle $|x| \leqq R'$.

5. *Expansion of $x^k/(t-x)$ in Terms of $\bar{F}_{k+m}(x)$.* In the expansion

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} \frac{x^{k+m}}{t^{m+1}},$$

which is valid for $|x| < |t|$, substitute the appropriate expansion (1) for each power of x:

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} \frac{1}{t^{m+1}} \sum_{s=0}^{\infty} \frac{(a+k+m)_s(-b)_s}{s!\ (1+k+m)_s} \bar{F}_{k+m+s}(x). \tag{5}$$

Rearrangement of this repeated series, assuming temporarily that this rearrangement is valid, gives

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} \left\{ \sum_{s=0}^{m} \frac{[a+k+m-1]_s(-b)_s}{s!\ \ [k+m]_s} \ \frac{1}{t^{m-s+1}} \right\} \bar{F}_{k+m}(x),$$

where $[a]_s = a(a-1)(a-2)\cdots(a-s+1)$ and $[a]_0 = 1$. We now define the polynomial $G_{m,k}(t)$ by the equation

$$G_{m,k}(t) = \sum_{s=0}^{m} \frac{[a+k+m-1]_s(-b)_s}{s!\ \ [k+m]_s} \cdot \frac{1}{t^{m-s+1}}, \quad (m \geqq 0);$$

$G_{m,k}(t)$ is a polynomial in $1/t$ of degree $m+1$. We then have the expansion

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} G_{m,k}(t) \cdot \bar{F}_{k+m}(x). \tag{6}$$

To justify the above rearrangement it is sufficient to prove the absolute convergence of the repeated series in (5). But for m large enough ($m > m_1$) so that $m+1-|k| > 0$, (5) is dominated by

$$\sum_{m=0}^{\infty} \frac{M}{|t|^{m+1}} \sum_{s=0}^{\infty} \left| \frac{(a+k+m)_s \cdot (-b)_s}{s!\ (1+k+m)_s} \right| \cdot |x|^{k+m+s}$$

$$\leqq M \sum_{m=0}^{\infty} \frac{|x|^{k+m}}{|t|^{m+1}} \sum_{s=0}^{\infty} \frac{(|a+k|+m)_s \cdot (|b|)_s}{s!\ (m+1-|k|)_s} \cdot |x|^s$$

$$= M \sum_{m=0}^{\infty} \frac{|x|^{k+m}}{|t|^{m+1}} F(|a+k|+m, |b|, m+1-|k|; |x|).$$

But by the arguments of Art. 3, $F(|a+k|+m, |b|, m+1-|k|; |x|)$ has an upper bound for all values of m greater than m_1 and for all values of x such that $|x| \leqq R' < 1$. We also surely have an upper bound when $m \leqq m_1$. Hence, letting M' be the greater of these upper bounds, (5) is dominated by

$$M.M'. \sum_{m=0}^{\infty} \frac{|x|^{k+m}}{|t|^{m+1}} ,$$

which converges absolutely when $|x| < |t|$, where $|x| \leqq R' < 1$. Consequently (5) is absolutely convergent and the expansion (6) is valid under these conditions.

We can readily establish the uniformity of the convergence of the expansion (6) divided by x^k throughout the regions

$$|t| \geqq R, \qquad |x| \leqq R' \qquad \text{where } R' < R,\ R' < 1.$$

When these inequalities are satisfied, (5), and consequently also (6), after dividing each by x^k, will be dominated by a convergent series of positive constants independent of x and t. Hence, the uniformity of the convergence of (6) divided by x^k follows from the test of Weierstrass.

Writing $\tilde{\bar{F}}_{k+m}(x) = x^{-k}\bar{F}_{k+m}(x) = x^m F_{k+m}(x)$, this result can be stated as

THEOREM I. *Let x and t be two complex variables such that $|x| \leqq R' < 1$ and $|t| \geqq R$, where $R' < R$; then the expansion*

$$\frac{1}{t-x} = \sum_{m=0}^{\infty} G_{m,k}(t)\tilde{\bar{F}}_{k+m}(x) \tag{7}$$

is valid, provided k is not a negative integer. If t be a variable point on the contour C formed by the circle $|t| = R$, and x any fixed point in the circle $|x| = R'$, then the expansion converges uniformly with regard to t.

6. *Expansion of $f(x)$ in Terms of $\bar{F}_{k+m}(x)$.* By means of (7) and Cauchy's integral formula, it is possible to get an expansion of $f(x)$ in a series of $\bar{F}_{k+m}(x)$, $(m = 0, 1, 2, \cdots)$, $f(x)$ being an arbitrary function of x analytic and one-valued inside and upon the contour C, formed by the circle $|x| = R \leqq 1$. We proceed as follows:

$$(8) \qquad f(x) = 1/2\pi i \int_c \frac{f(t)\,dt}{t - x}$$

$$= 1/2\pi i \int_c \left\{ \sum_{m=0}^{\infty} G_{m,k}(t) \,.\, \tilde{F}_{k+m}(x) \right\} f(t)\,dt,$$

that is,

$$(9) \qquad f(x) = \sum_{m=0}^{\infty} a_m \tilde{F}_{k+m}(x), \quad \text{where } a_m = 1/2\pi i \int_c G_{m,k}(t) f(t)\,dt.$$

This expansion converges uniformly with regard to x when x is in the circle $|x| = R' < 1$. That is, the expansion

$$x^k f(x) = \sum_{m=0}^{\infty} a_m \bar{F}_{k+m}(x)$$

is valid when $|x| \leqq R' < 1$. This gives

THEOREM II. *Let $f(x)$ be a function of x which is analytic and one-valued inside and upon the contour C formed by the circle $|x| = R \leqq 1$; then the expansion*

$$(10) \qquad x^k f(x) = \sum_{m=0}^{\infty} a_m \bar{F}_{k+m}(x), \quad \textit{where } a_m = 1/2\pi i \int_c G_{m,k}(t) f(t)\,dt,$$

is valid when x is in the circle $|x| = R'$, where $R' < R$, provided k is not a negative integer; and the expansion divided by the factor x^k converges uniformly with regard to x when $|x| \leqq R'$.

7. *Expansion of $f(x)$ in Terms of $G_{m,k}(x)$.* If x and t be two complex variables such that $|t| \leqq r$ and $1 > |x| \geqq r'$, where $r < r'$, then it follows from arguments analogous to those in Art. 5 that the expansion

$$(11) \qquad \frac{1}{x - t} = \sum_{m=0}^{\infty} \tilde{F}_{k+m}(t) \,.\, G_{m,k}(x)$$

is valid, provided k is not a negative integer; and if t be a variable point on the contour c formed by the circle $|t| = r$, and x any fixed point either on or outside the circle $|x| = r'$ but inside the circle $|x| = 1$, then the expansion (11) converges uniformly with regard to t.

Now if $f(x)$ is a function of x, one-valued and analytic, outside and upon the contour c, it can be shown by arguments analogous to those in Art. 6 to have an expansion in a series of $G_{m,k}(x)$, namely,

$$(12) \qquad f(x) = \sum_{m=0}^{\infty} a'_m G_{m,k}(x), \qquad \text{where } a'_m = 1/2\pi i \int_c \tilde{F}_{k+m}(t) f(t)\,dt,$$

which will be valid and converge uniformly with regard to x when x is either on or outside the circle $|x| = r'$ but within the circle $|x| = 1$, where $r < r'$, provided k is not a negative integer.

8. *Relation to Maclaurin's Expansion.* If the Maclaurin expansion of $f(x)$ is

$$f(x) = \sum_{m=0}^{\infty} b_m x^m,$$

while the expansion of Theorem II is

$$x^k f(x) = \sum_{m=0}^{\infty} a_m \bar{F}_{k+m}(x),$$

it is possible to find the relation between the coefficients b_m and a_m. We see that

$$\begin{aligned} a_m &= 1/2\pi i \int_C G_{m,k}(t) f(t)\, dt \\ &= 1/2\pi i \int_C G_{m,k}(t) \left\{ \sum_{n=0}^{\infty} b_n t^n \right\} dt \\ &= 1/2\pi i \sum_{n=0}^{\infty} b_n \int_C t^n \left\{ \sum_{\nu=0}^{m} \frac{[a+k+m-1]_\nu \; (-b)_\nu}{\nu!\; [k+m]_\nu} \cdot \frac{1}{t^{m-\nu+1}} \right\} dt \end{aligned}$$

$$(13) \qquad \begin{cases} a_m = \sum_{n=0}^{m} \dfrac{[a+k+m-1]_n (-b)_n}{n!\; [k+m]_n} \; b_{m-n}, & (m = 0, 1, 2, \cdots), \\ a_0 = b_0. \end{cases}$$

These formulae enable us to construct the expansion in Theorem II of a given function when the Maclaurin expansion of that function is given. It is clear that the expansion of a given function in a series of ordinary hypergeometric functions will have simple coefficients whenever the expressions in (13) take on simple forms.

9. *Analogue of Laurent's Theorem.* Let $f(x)$ be an analytic and one-valued function of x in the ring-shaped region defined by the inequalities

$$0 < r \leqq |x| \leqq R \leqq 1.$$

Let C and c be the contours formed by the circles $|x| = R$, $|x| = r$, and C' and c' the contours formed by the circles $|x| = R'$, $|x| = r'$, where $R > R'$ and $r < r'$.

Then, if x is any point in the ring-shaped region defined by the inequalities

$$r' \leqq |x| \leqq R',$$

we have from Cauchy's theorem that

$$f(x) = 1/2\pi i \int_C \frac{f(t)\,dt}{t-x} + 1/2\pi i \int_c \frac{f(t)\,dt}{x-t}, \tag{14}$$

where the integrations along the contours C and c are both taken in a counter-clockwise direction. For the first integral in (14), we have $|x| \leqq R'$ and $|t| = R$, where $R' < R \leqq 1$; hence, Theorem I applies and gives

$$\frac{1}{t-x} = \sum_{m=0}^{\infty} G_{m,k}(t) \cdot \tilde{F}_{k+m}(x).$$

Similarly for the second integral, where $|t| = r$, $|x| \geqq r'$, and $r < r'$, we have from (11)

$$\frac{1}{x-t} = \sum_{m=0}^{\infty} \tilde{F}_{k+m}(t) \cdot G_{m,k}(x).$$

Substituting these two expressions in (14), we have

$$f(x) = 1/2\pi i \int_C \left\{ \sum_{m=0}^{\infty} G_{m,k}(t) \cdot \tilde{F}_{k+m}(x) \right\} f(t)\,dt + 1/2\pi i \int_c \left\{ \sum_{m=0}^{\infty} \tilde{F}_{k+m}(t) \cdot G_{m,k}(x) \right\} f(t)\,dt.$$

Consequently, the expansion of $f(x)$ is in the form

$$f(x) = \sum_{m=0}^{\infty} a_m \tilde{F}_{k+m}(x) + \sum_{m=0}^{\infty} a'_m G_{m,k}(x),$$

where $a_m = 1/2\pi i \int_C G_{m,k}(t) f(t)\,dt$, $a'_m = 1/2\pi i \int_c \tilde{F}_{k+m}(t) f(t)\,dt$,

and is valid when $r' \leqq |x| \leqq R'$. This result may be stated as

THEOREM III. *If $f(x)$ is an analytic and one-valued function of x in the ring-shaped region defined by the inequalities*

$$0 < r \leqq |x| \leqq R \leqq 1,$$

the expansion

$$f(x) = \sum_{m=0}^{\infty} a_m \tilde{F}_{k+m}(x) + \sum_{m=0}^{\infty} a'_m G_{m,k}(x), \tag{15}$$

where

$$a_m = 1/2\pi i \int_C G_{m,k}(t) f(t)\,dt \quad \text{and} \quad a'_m = 1/2\pi i \int_c \tilde{F}_{k+m}(t) f(t)\,dt,$$

is valid and converges uniformly with regard to x when

$$r' \leqq |x| \leqq R',$$

where $r < r'$ and $R' < R$, provided k is not equal to a negative integer.

10. *Relation of the Analogue to the Actual Laurent Expansion.* If the Laurent expansion of $f(x)$ in the region.

$$r' \leqq |x| \leqq R'$$

is

$$f(x) = \sum_{m=0}^{\infty} b_m x^m + \sum_{m=1}^{\infty} b'_m x^{-m},$$

we can find the relation between the coefficients b_m, b'_m of this expansion and a_m, a'_m of (15). As before, in Art. 8, we find the value of a_m as given by the formulae (13); and we see that

$$\begin{aligned} a'_m &= 1/2\pi i \int_o \tilde{F}_{k+m}(t) f(t)\, dt \\ &= 1/2\pi i \int_o \tilde{F}_{k+m}(t) \left\{ \sum_{n=1}^{\infty} b'_n / t^n \right\} dt \\ &= 1/2\pi i \sum_{n=1}^{\infty} b'_n \int_o t^{-n} \tilde{F}_{k+m}(t)\, dt \\ &= 1/2\pi i \sum_{n=1}^{\infty} b'_n \int_o t^{-n} \left\{ \sum_{\nu=0}^{\infty} \frac{(a+k+m)_\nu (b)_\nu}{\nu!\ (1+k+m)_\nu} t^{m+\nu} \right\} dt. \end{aligned}$$

Hence, we have

$$(16) \qquad \begin{cases} a'_m = \sum_{n=0}^{\infty} \dfrac{(a+k+m)_n (b)_n}{n!\ (1+k+m)_n} b'_{m+n+1}, \\ a_m = \sum_{n=0}^{m} \dfrac{[a+k+m-1]_n (-b)_n}{n!\ [k+m]_n} b_{m-n}. \end{cases} \qquad (m = 0, 1, 2, \cdots).$$

11. *Recurrence Formulae for $\bar{F}_{k+n}(x)$.* Let the symbols F_0, F_1, and F_2, respectively, denote $F(a, b, c; x)$, $F(a+1, b, c+1; x)$, and $F(a+2, b, c+2; x)$, and the symbol F'_k the derivative of F_k with respect to x. From the four relations given by Gauss,

$$\begin{aligned} &(AA) \quad cF(a, b+1, c; x) - cF(a+1, b, c; x) \\ &\qquad -(a-b)xF(a+1, b+1, c+1; x) = 0, \\ &(BB) \quad c(c+1)F(a, b+1, c+1; x) - c(c+1)F(a, b, c; x) \\ &\qquad -a(c-b)xF(a+1, b+1, c+2; x) = 0, \end{aligned}$$

(17)

$$\begin{aligned} &(CC) \quad cF(a, b+1, c; x) - cF(a, b, c; x) \\ &\qquad -axF(a+1, b+1, c+1; x) = 0, \end{aligned}$$

$$(DD)\quad [c-2b+(b-a)x]F(a,b,c;x) + b(1-x)F(a,b+1,c;x)-(c-b)F(a,b-1,c;x)=0,$$

one can easily derive the following formulae, from which in turn we can derive recurrence relations for $\bar{F}_{k+n}(x)$.

$$(A)\quad c(c+1)F_0-(c+1)[c+(a-b+1)x]F_1 +(a+1)(c-b+1)xF_2=0,$$

$$(B)\quad acF_0-c(1-x)F'_0+a(b-c)F_1=0,$$

$$(C)\quad c(a+1)(c+1)F_0-(c+1)(1-x)[c+(a-b+1)x]F'_1 +(a+1)(b-c-1)(c-bx)F_2=0,$$

$$(D)\quad cF'_0-abF_1-axF'_1=0,$$

(18)

$$(E)\quad c(c+1)(c+1-bx)F'_0 -ab(c+1)[c+1+(a-b+1)x]F_1 -a(a+1)(c-b+1)x^2F'_2=0,$$

$$(F)\quad cF_0-(c-bx)F_1-x(1-x)F'_1=0,$$

$$(G)\quad abcF_0-c(c-bx)F'_0-a(b-c)xF'_1=0,$$

$$(H)\quad c(c+1)(1-x)F'_0-a(c+1)[b+(a-b+1)x]F_1 +a(a+1)(c-b+1)xF_2=0.$$

To derive (A): replace b by $b-1$ in (AA); a, b, respectively, by $a+1$, $b+1$ in (BB); and eliminate $F(a+1, b-1, c; x)$ between the two equations. The relation (B) is derived from the three relations (CC), (DD), and (CC) with b replaced by $b-1$, by eliminating $F(a, b+1, c; x)$ and $F(a, b-1, c; x)$. To get (C), replace a, c by $a+1, c+1$ in (B) and eliminate F_1 between the result and (A). We get (D) from (CC) by replacing a, c, respectively, by $a+1, c+1$.

To get (E): replace a, c by $a+1, c+1$ in (CC); a, b, c by $a+1, b+1, c+1$ in (A); and eliminate $F(a+2, b+1, c+2; x)$. Now get (F) from (B) and (D) by eliminating F'_0; and (G) from (B) and (D) by eliminating F_1. Finally, eliminate F_0 between (A) and (B) to get (H).

The corresponding recurrence relations for $F_{k+n}=F_{k+n}(x)=F(a+k+n, b, 1+k+n; x)$ are derived from the formulae (18) by replacing a, b, c, respectively, by $a+k+n-1, b, k+n$. We shall not take the space to write down these resulting formulae for $F_{k+n}(x)$, but we shall refer to them as (A′), (B′), · · ·, and (H′).

The recurrence formulae for $\bar{F}_{k+n}=\bar{F}_{k+n}(x)=x^{k+n}F(a+k+n, b, 1+k+n; x)$ corresponding to those for F_{k+n}, and derived from them by means of the relations

(19) $$F_{k+n} = x^{-k-n}\bar{F}_{k+n}, \quad F'_{k+n} = x^{-k-n}\bar{F}'_{k+n} - (k+n)x^{-k-n-1}\bar{F}_{k+n},$$

are

$$(\alpha)\quad (k+n)(k+n+1)x\bar{F}_{k+n-1} - (k+n+1)[k+n+(a-b+k+n)x]\bar{F}_{k+n} + (a+k+n)(1-b+k+n)\bar{F}_{k+n+1} = 0,$$

$$(\beta)\quad (k+n)(k+n-1+ax)\bar{F}_{k+n-1} - (k+n)x(1-x)\bar{F}'_{k+n-1} + (a+k+n-1)(b-k-n)\bar{F}_{k+n} = 0,$$

$$(\gamma)\quad (k+n)(k+n+1)(k+n+ax)\bar{F}_{k+n-1} - (k+n+1)(1-x)[k+n+(a-b+k+n)x]\bar{F}'_{k+n} - (a+k+n)(1-b+k+n)(k+n-b)\bar{F}_{k+n+1} = 0,$$

$$(\delta)\quad (k+n)x\bar{F}'_{k+n-1} + a(k+n-b)\bar{F}_{k+n} - (k+n-1+ax)\bar{F}'_{k+n} = 0,$$

(20)

$$(\epsilon)\quad (k+n)\bar{F}_{k+n-1} + (b-k-n)\bar{F}_{k+n} - (1-x)\bar{F}'_{k+n} = 0,$$

$$(\zeta)\quad a(k+n)\bar{F}_{k+n-1} + (k+n)x\bar{F}'_{k+n-1} - (a+k+n-1)\bar{F}'_{k+n} = 0,$$

$$(\eta)\quad (k+n)(k+n+1)x^2\bar{F}'_{k+n-1} + a(k+n+1)[k+n-1+(a-b+k+n)x]\bar{F}_{k+n} - (a+k+n)(k+n-1+ax)\bar{F}'_{k+n+1} = 0,$$

$$(\theta)\quad (k+n)(k+n+1)x^2(1-x)\bar{F}'_{k+n-1} - (k+n+1)\{(k+n)(k+n-1) + a(k+n-1+b)x + a(a-b+k+n)x^2\}\bar{F}_{k+n} + (a+k+n)(1-b+k+n)(k+n-1+ax)\bar{F}_{k+n+1} = 0.$$

Relations (α), (β), and (ϵ) follow directly from (A′), (B′), and (F′). To get (γ), transform (C′) by means of (19) and eliminate $\bar{F}_{k+n}$ between this result and (α). To get (δ), transform (D′) by means of (19) and eliminate $\bar{F}_{k+n-1}$ between this and (β). To get (ζ), eliminate $\bar{F}_{k+n}$ between (δ) and (ϵ). To get (η), replace n by $n+1$ in (ζ) and eliminate $\bar{F}'_{k+n}$ between this and (δ). Finally, (θ) is derived from (α) and (β) by eliminating $\bar{F}_{k+n-1}$.

12. *Function Dominating $G_{m,k}(t)$.* It is possible to construct a simple dominating function for the polynomial $G_{m,k}(t)$ which was defined as the coefficient of $\bar{F}_{k+m}(x)$ in the expansion (6). Assuming throughout that k is not a negative integer, we have

$$|G_{m,k}(t)| \leqq \frac{1}{|t|^{m+1}} \sum_{s=0}^{m} \left|\frac{[a+k+m-1]_s}{[k+m]_s}\right| \left|\frac{(|b|)_s}{s!}\right| |t|^s.$$

By means of the asymptotic formula for the gamma function, it can be shown

that for m large enough ($m \geqq m_1$) the coefficients

$$\left| \frac{(a+k+m-1)(a+k+m-2)\cdots(a+k+m-s)}{(k+m)(k+m-1)\cdots(k+m-s+1)} \right|, \qquad (s=1, 2, \cdots, m),$$

remain less than $M_1 m^{|a-1|}$ for all values of $m > m_1$, and for all values of a, where M_1 is some positive constant. Then

$$|G_{m,k}(t)| < M_1 \frac{m^{|a-1|}}{|t|^{m+1}} \sum_{s=0}^{m} \frac{(|b|)_s}{s!} |t|^s$$

$$< M_1 \frac{m^{|a-1|}}{|t|^{m+1}} \sum_{s=0}^{\infty} \frac{(|b|)_s}{s!} |t|^s = M_1 \frac{m^{|a-1|}}{|t|^{m+1}} [1-|t|]^{-|b|}.$$

But when $|t| \leqq R'' < 1$, the quantity $[1-|t|]^{-|b|}$ has an upper bound M_2 for all values of m; and surely for all values of m less than m_1, we have

$$|G_{m,k}(t)| < \frac{M_3}{|t|^{m+1}}, \text{ where } M_3 \text{ is some positive constant.}$$

Let M_4 be some positive constant such that $M_3 > M_4 m^{|a-1|}$, where $m < m_1$. Let M' be the greater of the quantities $(M_1 . M_2)$ and M_4; then

$$|G_{m,k}(t)| < M' \frac{m^{|a-1|}}{|t|^{m+1}} \tag{21}$$

for all values of m, and for all values of t such that $|t| \leqq R'' < 1$. This expression and the corresponding one for $|\bar{F}_{k+m}(x)|$, of Art. 3, enable us to dominate the expansion (6) by the series

$$\sum_{m=0}^{\infty} \frac{|x|^{k+m}}{|t|^{m+1}} m^{|a-1|} . M . M'.$$

By the ratio test, this series, and consequently also (6), is seen to be absolutely convergent when $|x| < |t| \leqq R'' < 1$. Furthermore, the uniform convergence of (6) divided by the factor x^k can now be established throughout the regions

$$|x| \leqq R', \quad |t| \geqq R, \quad \text{where } R' < R \leqq R'' < 1, \quad |t| \leqq R'' < 1;$$

for then (6) is dominated by

$$\sum_{m=0}^{\infty} \frac{(R')^m}{R^{m+1}} m^{|a-1|} M . M',$$

which is an absolutely convergent series of positive constants. Hence, the uniform convergence of (6) divided by x^k follows from the test of Weierstrass, provided that k is not equal to a negative integer.

13. *Recurrence Formulae for the Polynomials* $G_{n,k}(t)$. We shall now derive a few recurrence relations satisfied by $G_{n,k} = G_{n,k}(t)$, the symbol $G'_{n,k}$ denoting the derivative of $G_{n,k}$ with respect to t:

$$
\begin{aligned}
&(1-a-k-n)(k-b+n)G_{n-1,k} \\
&\quad +(k+n)[n+1+(a+k-2)t]G_{n,k} \\
&\qquad +(k+n)t(1-t)G'_{n,k} = h_n, \\
&(a+k+n-1)(k-b+n)tG'_{n-1,k} \\
&\quad -(k+n)[(n+1)k+(b-n)(a+k-2)t]G_{n,k} \\
&\qquad -(k+n)t[k+(n-b)t]G'_{n,k} = (n-b)h_n,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&[k+(n-b)t]G_{n-1,k} - t(1-t)G'_{n-1,k} - (k+n)tG_{n,k} \\
&\qquad = -\frac{h_n}{a+k+n-1}, \\
&(a+k+n-1)(k-b+n)G_{n-1,k} \\
&\quad -(k+n)[k+n+1+(a+k-b+n-1)t]G_{n,k} \\
&\qquad +(k+n)(k+n+1)tG_{n+1,k} = -\frac{b+1}{n+1}\,h_n,
\end{aligned}
$$

$$
\text{where } h_n = \frac{\Gamma(a+k+n)\Gamma(-b+n)\Gamma(1+k)}{\Gamma(a+k-1)\Gamma(-b)\Gamma(k+n)n!},
$$

these formulae being valid for $n \geqq 1$. It may be well to note that for $n = 1$, these relations may be put into different forms because of the existence of two-term relations connecting $G_{0,k}$, $G'_{0,k}$, $G_{1,k}$, and $G'_{1,k}$.

To derive the first formula in (22), we note that $G_{n,k}(t)$ is equal to the first $n+1$ terms in the expansion of $t^{-n-1}F(1-a-k-n, -b, -k-n; t)$ in powers of t. Replacing, respectively, $a+k+n-1$, b, and $k+n$ in (18-B′) by $1-a-k-n$, $-b$, and $-k-n$, we have

$$
\begin{aligned}
&(a+k+n-1)(k+n)F(1-a-k-n, -b, -k-n; t) \\
&\quad +(k+n)(1-t)F'(1-a-k-n, -b, -k-n; t) \\
&\quad +(1-a-k-n)(k-b+n) \\
&\qquad \times F(2-a-k-n, -b, 1-k-n; t) = 0.
\end{aligned}
$$

If in the first member of this equation we replace $F(1-a-k-n, -b, -k-n; t)$, $F'(1-a-k-n, -b, -k-n; t)$, and $F(2-a-k-n, -b, 1-k-n; t)$, respectively, by $t^{n+1}G_{n,k}(t)$, $[t^{n+1}G'_{n,k}(t) + (n+1)t^n \cdot G_{n,k}(t)]$, and $t^nG_{n-1,k}(t)$, we shall have the first recurrence relation in (22) with the second member, h_n, still undetermined. Its value, however, is readily found by substituting the polynomials $G_{n-1,k}$, $G_{n,k}$, and $G'_{n,k}$ into the given relation.

To derive the second formula in (22), we get in similar fashion a recurrence relation, except for undetermined second member, connecting $G_{n-1,k}$, $G'_{n-1,k}$, $G_{n,k}$, and $G'_{n,k}$, from (18-D′) and eliminate $G_{n-1,k}$ between this and the first formula in (22). The second member is then determined in the same fashion as above.

Now the third result in (22) readily follows from the first two by elimination of $G'_{n,k}$, and the last one is derived by replacing n by $n+1$ in the third and then eliminating $G'_{n,k}$ between this result and the first relation.

14. *The Differential Equation Satisfied by* $G_{n,k}(t)$. By differentiating the first recurrence relation in (22) and eliminating $G'_{n-1,k}$ between this result and the second relation in (22), we get the differential equation satisfied by $G_{n,k}(t)$:

$$(23)\quad t(1-t)y'' + [2-k+n+(a+b+k-n-4)t]y' - \left[\frac{(n+1)k}{t} + (a+k-2)(b-1-n)\right] y = \frac{n-b}{k+n}\cdot\frac{h_n}{t},$$

where $$h_n = \frac{\Gamma(a+k+n)\Gamma(-b+n)\Gamma(1+k)}{\Gamma(a+k-1)\Gamma(-b)\Gamma(k+n)\ n!}.$$

15. *Orthogonality Properties.* We shall now obtain the integral formulae

$$(24)\quad \begin{aligned} &\int_L \tilde{F}_{k+m}(x)\,.\,\tilde{F}_{k+n}(x)\,dx = 0, && (m=n \text{ and } m\neq n),\\ &\int_L G_{m,k}(x)\,.\,G_{n,k}(x)\,dx = 0, && (m=n \text{ and } m\neq n),\\ &\int_L \tilde{F}_{k+n}(x)\,.\,G_{n,k}(x)\,dx = 2\pi i\kappa, && (L \text{ enclosing origin}),\\ &\qquad\qquad = 0, && (L \text{ not enclosing origin}),\\ &\int_L \tilde{F}_{k+m}(x)\,.\,G_{n,k}(x)\,dx = 0, && (m\neq n), \end{aligned}$$

where the integration is in a positive direction along L, any closed contour whatever, not passing through the origin and lying within the circle $|x|=1$. Since $\tilde{F}_{k+m}(x)$ is single-valued and uniformly continuous in the region enclosed by L, the first formula follows immediately from Cauchy's theorem. Since the residue of $G_{m,k}(x)\,.\,G_{n,k}(x)$ at $x=0$ is zero, the second follows readily. We also have

$$\int_L \tilde{F}_{k+n}(x)\,.\,G_{n,k}(x)\,dx = \int_L dx/x + \sum_{\nu=0}^{\infty}\int_L a_\nu x^\nu dx,$$

which is equal to $2\pi i\kappa$ or zero according as L does or does not enclose the origin, where κ is the excess of the number of positive circuits of the contour around the origin over the number of negative circuits.

In proving the fourth formula, we note that it is true when $m > n$ because the expansion of the integrand in x^{-1} then contains no term in x^{-1}. When $m < n$, say $m = n - s$, $(s = 1, 2, 3, \cdots, n)$, the residue of the integrand at $x = 0$ is

$$2\pi i \frac{(a+k+n-1)(a+k+n-2)\cdots(a+k+n-s)}{s!\,(k+n)(k+n-1)\cdots(k+n-s+1)} \times \{[b+(-b)]\cdot[(b+1)+(-b+1)]\cdots \cdot[(b+s-1)+(-b+s-1)]\},$$

which is zero for all values of s because of the zero factor $[b+(-b)]$. Hence, the fourth formula will be true whether or not L encloses the origin.

By means of these integral formulae (24), the expansions of a function $f(x)$ in series of the forms (9), (12), and (15) can be obtained directly, assuming $f(x)$ to be analytic in the circle $|x| = R \leqq 1$. For example, if we assume $f(x)$ to be expanded in a series of the form

$$f(x) = \sum_{m=0}^{\infty} a_m \tilde{F}_{k+m}(x), \tag{9}$$

the coefficients a_m are readily determined by means of the third and fourth formulae in (24). For, if we multiply both members of the equation (9) by $G_{m,k}(x)$ and then integrate each term with respect to x along a contour C enclosing the origin and lying within the unit circle, we have:

$$\int_C G_{m,k}(x) f(x)\,dx = a_m \int_C G_{m,k}(x)\cdot\tilde{F}_{k+m}(x)\,dx,$$

since by the fourth relation in (24) all the remaining integrals in the second member vanish. We, therefore, have the value of any coefficient a_m in (9), namely:

$$a_m = 1/2\pi i \int_C G_{m,k}(t) f(t)\,dt.$$

Similarly, the coefficients in the expansion (15) are directly determined by means of all four formulae in (24).

16. *The Expansion in Special Cases.* *When* $k = 0$, the above developed expansion theory, Arts. 2 to 15 inclusive, will hold in every respect, expansions now being in terms of

$$\bar{F}_n(x) = x^n F(a+n,\ b,\ 1+n;\ x).$$

We may note that there will be no restrictions on a or b.

It is interesting to notice that for $k = 0$ the differential equation (23) satisfied by $G_{n,0}(t)$, is a non-homogeneous differential equation having $(n-b)h_n/t(k+n)$ in the second member, which in turn is a hypergeometric differential equation when the second member is zero. This second member is zero either when $a = 1$ or when $b = 0$.

When $k = 0$ and $a = 1$, our expansions are in terms of

$$\bar{F}_n(x) = x^n F(1+n,\ b,\ 1+n;\ x) = x^n(1-x)^{-b}$$

and

$$G_n(t) = t^{-n-1} \sum_{\nu=0}^{n} \binom{b}{\nu} (-t)^\nu,$$

where $\bar{F}_n(x)$ and $G_n(t)$ are both solutions of hypergeometric differential equations simply related to each other, namely:

$$x(1-x)y'' + [1-n-(b-n+2)x]y' - (b-n)y = 0,$$
$$t(1-t)y'' + [2+n-(-b+n+3)t]y' - (n+1-b)y = 0.$$

Finally, when $k = 0$ and $b = 0$, the expansions are in terms of $\bar{F}_n(x) = x^n F(1+n,\ 0,\ 1+n;\ x) = x^n$ and $G_n(t) = t^{-n-1}$, again both being solutions of hypergeometric differential equations. We see, then, that the expansion of Theorem II reduces in this special case to the Cauchy-Taylor expansion, namely:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{where } a_n = 1/2\pi i \int_c \frac{f(t)\,dt}{t^{n+1}}.$$

Similarly, the expansion of Theorem III reduces to the ordinary Laurent expansion:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a'_n x^{-n-1},$$

where $a_n = 1/2\pi i \int_c \frac{f(t)\,dt}{t^{n+1}}$, and $a'_n = 1/2\pi i \int_c t^n f(t)\,dt.$

There exists no expansion of $\tilde{F}_{k+m}(x)$ in terms of other hypergeometric functions, $\tilde{F}_{k+m+s}(x)$, $(s = 1,\ 2,\ 3,\ \cdots)$; but replacing n by $n+1$ in the recurrence relation (20-a) and by applying this relation repeatedly, we can get an expansion of $\tilde{F}_{k+m}(x)$ in terms of $x^{-s-1}\tilde{F}_{k+m+2s+1}(x)$, namely:

$$\tilde{F}_{k+m}(x) = \sum_{s=0}^{\infty} \gamma_s [k+m+2s+1+(a-b+k+m+2s+1)x] \times \frac{(2)^{2s}}{x^{s+1}}\ \tilde{F}_{k+m+2s+1}(x),$$

$$\text{where } \gamma_s = (-1)^s \frac{\left(\frac{a+k+n-1}{2}+1\right)_s \left(\frac{k-b+n}{2}+1\right)_s}{(k+n+1)_{2s+1}}.$$

By showing that $\underset{s\to\infty}{L} R_s \to 0$, where R_s is the remainder after $s+1$ terms, we can prove this expansion to be valid when $|x| \leqq R' < 1$.

III. Expansions in Terms of Generalized Hypergeometric Functions.

17. *General Remarks.* The entire expansion theory developed above is now to be extended to the generalized hypergeometric functions of Goursat * and Pochhammer,† which satisfy a homogeneous linear differential equation of order $q+1$, two of whose particular integrals are

$$ {}_pF_q(a_1, a_2, \cdots, a_p; \rho_1, \rho_2, \cdots, \rho_q; x) = \sum_{s=0}^{\infty} \frac{(a_1)_s \cdot (a_2)_s \cdots (a_p)_s}{(\rho_1)_s \cdot (\rho_2)_s \cdots (\rho_q)_s} \cdot \frac{x^s}{s!} $$

and

$$ x^{1-\rho_1} {}_pF_q(a_1-\rho_1+1, a_2-\rho_1+1, \cdots, a_p-\rho_1+1; 2-\rho_1, \rho_2-\rho_1+1, \rho_3-\rho_1+1, \cdots, \rho_q-\rho_1+1; x) $$

$$ (25) \quad = x^{1-\rho_1} \sum_{s=0}^{\infty} \frac{(a_1-\rho_1+1)_s \cdot (a_2-\rho_1+1)_s \cdots (a_p-\rho_1+1)_s}{(2-\rho_1)_s \cdot (\rho_2-\rho_1+1)_s \cdot (\rho_3-\rho_1+1)_s \cdots (\rho_q-\rho_1+1)_s} \times \frac{x^s}{s!}, \quad (p \geqq 0,\ q \geqq 1), $$

where $(a)_s = a(a+1)(a+2) \cdots (a+s-1)$, $(a)_0 = 1$, and where $(a_p-\rho_1+1)_s = 1$ for $p = 0$. We may note that the order of the differential equation depends upon the number q and not on p.

These series all diverge when $p > q+1$. It is convenient to consider separately the two general cases: (1) $p = q+1$ and (2) $p < q+1$. *When* $p = q+1$, the differential equation has two finite singularities, namely, 0 and 1, and the series in (25) converges within the unit circle $|x| = 1$. However, *when* $p < q+1$, the differential equation has only one finite singularity, $x = 0$, and the series in (25) converges for all finite values of x.

18. *Expansions in Terms of Generalized Hypergeometric Functions, Valid Within the Unit Circle: Case,* $p = q+1$. An arbitrary analytic function of the complex variable x is now to be expanded in terms of

* *Loc. cit.*
† *Loc. cit.*

$$(26)\qquad \overline{\mathfrak{F}}_{k+m}(x) = x^{k+m}\mathfrak{F}_{k+m}(x)$$
$$= x^{k+m}\sum_{s=0}^{\infty}\frac{(a_1+k+m)_s\cdot(a_2+k+m)_s\cdots(a_{p-1}+k+m)_s\cdot(a_p)_s}{(1+k+m)_s\cdot(\rho_2+k+m)_s\cdots(\rho_{p-1}+k+m)_s\quad s!}x^s$$
$$(m=0,\ 1,\ 2,\ \cdots),\quad (p\geqq 2,\ \rho_1=1),$$

which is the form that the particular integral (25) changes into when we replace q, ρ_1, and a_p, respectively, by $p-1$, $1-k-m$, and a_p-k-m. Here $a_1, a_2, \cdots, a_p, \rho_2, \rho_3, \cdots, \rho_{p-1}$, and k are any fixed numbers, real or complex, except that none of the quantities $1+k$, ρ_2+k, ρ_3+k, $\cdots$, $\rho_{p-2}+k$, and $\rho_{p-1}+k$ is permitted to be a negative integer or zero. The expansions in terms of $\bar{F}_{k+m}(x) = x^{k+m}{}_2F_1(a+k+m,\ b;\ 1+k+m;\ x)$, investigated in Section II, are a special case of this, namely, when $p=2$.

A dominating expression for $\overline{\mathfrak{F}}_{k+m}(x)$ can be developed by means of an argument similar to that used in Art. 3 in connection with $\bar{F}_{k+m}(x)$. Here $\mathfrak{F}_{k+m}(x)$ can be shown to have an upper bound M for all values of m and for all values of x such that $|x| \leqq R' < 1$, since the series for $\mathfrak{F}_{k+m}(x)$ converges absolutely and uniformly for any finite value of m and for all values of x such that $|x| \leqq R' < 1$. We can choose m large enough ($m \geqq m_1$) so that any particular one of the $p-1$ sets of ratios

$$\frac{|a_1+k+m+\nu|}{|1+k+m+\nu|}\ ,\qquad \frac{|a_i+k+m+\nu|}{|\rho_i+k+m+\nu|}\ ,\qquad (i=2,\ 3,\ \cdots,\ p-1)$$
$$(\nu=0,\ 1,\ 2,\ \cdots),$$

will *either* be $\leqq 1$ for all values of m greater than or equal to m_1, *or* be $\geqq 1$ for all $m \geqq m_1$, and moreover that these ratios approach 1 monotonically as m increases beyond m_1. Let us suppose that σ of these $p-1$ sets are $\leqq 1$, while the remaining $p-1-\sigma$ sets are $\geqq 1$ when $m \geqq m_1$. Now if in the expansion of $\mathfrak{F}_{k+m_1}(x)$,

$$|\mathfrak{F}_{k+m_1}(x)| \leqq \sum_{s=0}^{\infty}\frac{|a_1+k+m_1|_s\cdot|a_2+k+m_1|_s\cdots|a_{p-1}+k+m_1|_s\cdot|a_p|_s}{|1+k+m_1|_s\cdot|\rho_2+k+m_1|_s\cdots|\rho_{p-1}+k+m_1|_s\cdot s!}|x|^s,$$

we replace each of the ratios in these σ sets of ratios $\leqq 1$ by 1, the series in the second member will be dominated by a series which is absolutely and uniformly convergent when $|x| \leqq R' < 1$; hence, the dominating series, and also $|\mathfrak{F}_{k+m_1}(x)|$, will have an upper bound M_1 for $|x| \leqq R' < 1$. If in this dominating series we replace m_1 by any value of m (say m_2) which is greater than m_1, we get a new series which is term by term less than the series in m_1. We readily see, therefore, that $|\mathfrak{F}_{k+m_2}(x)|$, $(m_2 > m_1)$, has an upper bound M_2.

Also $|\mathfrak{F}_{k+m}(x)|$ has an upper bound in the region $|x| \leqq R' < 1$ for each value of m less than m_1. Let M_3 be the largest of these. It is evident

that these results hold equally well when σ is equal to zero or $p-1$. Then if M is the greatest of the quantities M_1, M_2, and M_3, we have

$$|\,\bar{\mathfrak{F}}_{k+m}(x)\,| \leqq |\,x\,|^{k+m} M \tag{27}$$

for *all* values of m and for all values of x such that $|\,x\,| \leqq R' < 1$.

By the same method used (Art. 2) in finding the expansion of x^{k+m} in terms of $\bar{F}_{k+m}(x)$, we show that

$$x^{k+m} = \sum_{s=0}^{\infty} \frac{(1+k+m)_s \cdot (\rho_2+k+m)_s \cdots (\rho_{p-1}+k+m)_s \cdot s!}{(a_1+k+m)_s \cdot (a_2+k+m)_s \cdots (a_{p-1}+k+m)_s \cdot (-a_p)_s} \bar{\mathfrak{F}}_{k+m+s}(x). \tag{28}$$

By means of the expression in (27) dominating $|\,\bar{\mathfrak{F}}_{k+m}(x)\,|$, this expansion is readily proved to be absolutely convergent when $|\,x\,| \leqq R' < 1$; and after dividing out the factor x^k, it is seen to be uniformly convergent when $|\,x\,| \leqq R' < 1$.

Similarly as in Art. 5, we have, assuming $|\,x\,| < |\,t\,|$,

$$\frac{x^k}{t-x} = \sum_{m=0}^{\infty} \frac{x^{k+m}}{t^{m+1}}$$

$$= \sum_{m=0}^{\infty} \frac{1}{t^{m+1}} \sum_{s=0}^{\infty} \frac{(a_1+k+m)_s (a_2+k+m)_s \cdots (a_{p-1}+k+m)_s (-a_p)_s}{(1+k+m)_s \cdot (\rho_2+k+m)_s \cdots (\rho_{p-1}+k+m)_s \cdot s!} \bar{\mathfrak{F}}_{k+m+s}(x)$$

$$= \sum_{m=0}^{\infty} \mathfrak{g}_{m,k}(t) \cdot \bar{\mathfrak{F}}_{k+m}(x), \tag{29}$$

where $\mathfrak{g}_{m,k}(t)$, the coefficient of $\bar{\mathfrak{F}}_{k+m}(x)$, is a polynomial of degree $m+1$ in t^{-1}, defined by the equation

$$\mathfrak{g}_{m,k}(t) = \sum_{s=0}^{m} \frac{[a_1+k+m-1]_s \cdot [a_2+k+m-1]_s \cdots [a_{p-1}+k+m-1]_s \cdot (-a_p)_s}{[k+m]_s \cdot [\rho_2+k+m-1]_s \cdots [\rho_{p-1}+k+m-1]_s \cdot s!} \frac{1}{t^{m-s+1}}, \tag{30}$$

where $[a]_s = a(a-1)(a-2)\cdots(a-s+1)$, $[a]_0 = 1$.

With arguments similar to those used in Art. 5, we justify the rearrangement of the above repeated series, prove the expansion (29) to be valid and converge absolutely when $|\,x\,| < |\,t\,|$ and $|\,x\,| \leqq R' < 1$, and prove that (29) divided by x^k converges uniformly when $|\,x\,| \leqq R'$, $|\,t\,| \geqq R$, where $R' < R \leqq 1$.

Writing $\tilde{\mathfrak{F}}_{k+m}(x) = x^m \mathfrak{F}_{k+m}(x)$, we can now state for the generalized hypergeometric functions (case $p = q+1$) theorems corresponding to Theorems I, II, and III, keeping in each the restriction that none of the quantities $1+k$, ρ_2+k, ρ_3+k, $\cdots$, $\rho_{p-1}+k$ may be a negative integer or zero:

THEOREM IV. *Let x and t be two complex variables such that*

$$|x| \leqq R' \text{ and } |t| \geqq R, \text{ where } R' < R \leqq 1,$$

then the expansion

$$\frac{1}{t-x} = \sum_{m=0}^{\infty} \mathfrak{G}_{m,k}(t)\, \tilde{\mathfrak{F}}_{k+m}(x) \tag{31}$$

is valid. If t be a variable point on the contour C formed by the circle $|t| = R$, and x any fixed point in the circle $|x| = R'$, then the expansion converges uniformly with regard to t.

THEOREM V. *Let $f(x)$ be a function of x which is analytic and one-valued inside and upon the contour C formed by the circle $|x| = R \leqq 1$, then the expansion*

$$x^k f(x) = \sum_{m=0}^{\infty} a_m \bar{\mathfrak{F}}_{k+m}(x), \text{ where } a_m = 1/2\pi i \int_C \mathfrak{G}_{m,k}(t) f(t)\, dt, \tag{32}$$

is valid when $|x| \leqq R' < R \leqq 1$; and the expansion divided by the factor x^k converges uniformly with regard to x when $|x| \leqq R'$.

THEOREM VI. *If $f(x)$ is an analytic and one-valued function of x in the ring-shaped region defined by the inequalities*

$$0 < r \leqq |x| \leqq R \leqq 1,$$

the expansion

$$f(x) = \sum_{m=0}^{\infty} a_m \tilde{\mathfrak{F}}_{k+m}(x) + \sum_{m=0}^{\infty} a'_m \mathfrak{G}_{m,k}(x), \tag{33}$$

where $a_m = 1/2\pi i \int_C \mathfrak{G}_{m,k}(t) f(t)\, dt$ *and* $a'_m = 1/2\pi i \int_c \tilde{\mathfrak{F}}_{k+m}(t) f(t)\, dt$,

is valid and converges uniformly with regard to x when

$$r' \leqq |x| \leqq R',$$

where $r' > r$ and $R' < R$, and where C and c are the contours formed by the circles $|x| = R'$ and $|x| = r'$, respectively.

The formulae corresponding to (13) of Art. 8, and giving the relation between the coefficients b_m of *Maclaurin's expansion,*

$$f(x) = \sum_{m=0}^{\infty} b_m x^m,$$

and the coefficients a_m of the expansion of Theorem V, are

$$(34)\quad \begin{cases} a_m = \sum_{n=0}^{m} \dfrac{[a_1+k+m-1]_n \cdot [a_2+k+m-1]_n \cdots [a_{p-1}+k+m-1]_n \cdot (-a_p)_n}{[k+m]_n \cdot [\rho_2+k+m-1]_n \cdots [\rho_{p-1}+k+m-1]_n \cdot n!}\, b_{m-n}, \\ a_0 = b_0, \qquad (m = 0, 1, 2, \cdots). \end{cases}$$

Similarly, the formulae corresponding to (16) of Art. 10, giving the relation between the coefficients b_m, b'_m of *Laurent's* actual expansion,

$$f(x) = \sum_{m=0}^{\infty} b_m x^m + \sum_{m=1}^{\infty} b'_m x^{-m},$$

and the expansion of Theorem VI, will be

$$(35)\quad \begin{cases} a'_m = \sum_{n=0}^{\infty} \dfrac{(a_1+k+m)_n \cdot (a_2+k+m)_n \cdots (a_{p-1}+k+m)_n \cdot (a_p)_n}{(1+k+m)_n \cdot (\rho_2+k+m)_n \cdots (\rho_{p-1}+k+m)_n \quad n!}\, b'_{m+n+1}, \\ a_m = \sum_{n=0}^{m} \dfrac{[a_1+k+m-1]_n \cdot [a_2+k+m-1]_n \cdots [a_{p-1}+k+m-1]_n \cdot (-a_p)_n}{[k+m]_n \cdot [\rho_2+k+m-1]_n \cdots [\rho_{p-1}+k+m-1]_n \cdot n!}\, b_{m-n}, \\ \qquad\qquad (m = 0, 1, 2, \cdots). \end{cases}$$

Orthogonality properties corresponding to those of Art. 15 hold:

$$(36)\quad \begin{aligned} &\int_L \tilde{\mathfrak{F}}_{k+m}(x) \cdot \tilde{\mathfrak{F}}_{k+n}(x)\, dx = 0, && (m = n \text{ and } m \neq n), \\ &\int_L \mathfrak{G}_{m,k}(x) \cdot \mathfrak{G}_{n,k}(x)\, dx = 0, && (m = n \text{ and } m \neq n), \\ &\int_L \tilde{\mathfrak{F}}_{k+n}(x) \cdot \mathfrak{G}_{n,k}(x)\, dx = 2\pi i\kappa, && (L \text{ enclosing origin}), \\ &\qquad\qquad = 0, && (L \text{ not enclosing origin}), \\ &\int_L \tilde{\mathfrak{F}}_{k+m}(x) \cdot \mathfrak{G}_{n,k}(x)\, dx = 0, && (m \neq n), \end{aligned}$$

where L and κ have the same meanings as in Art. 15.

19. *Expansions in Terms of Generalized Hypergeometric Functions, Valid in Any Finite Region: Case $p < q + 1$.* Arbitrary functions of x, analytic in any finite region, are now to be expanded in terms of

$$(37)\quad \begin{aligned} \overline{\mathfrak{F}}_{k+m}(x) &= x^{k+m}\, \mathfrak{F}_{k+m}(x) \\ &= x^{k+m} \sum_{s=0}^{\infty} \frac{(a_1+k+m)_s \cdot (a_2+k+m)_s \cdots (a_p+k+m)_s}{(1+k+m)_s \cdot (\rho_2+k+m)_s \cdots (\rho_q+k+m)_s} \cdot \frac{x^s}{s!}, \\ &\qquad (m = 0, 1, 2, \cdots), \quad (p \leqq q,\ q \geqq 1,\ \rho_1 = 1), \end{aligned}$$

where $(a_p + k + m)_s = 1$ for $p = 0$, which is the form assumed by the particular integral (25) when we replace ρ_1 by $1 - k - m$. Again, $a_1, a_2, \cdots,$

a_p, ρ_2, ρ_3, $\cdots$, ρ_q, and κ are any fixed numbers, real or complex, except that none of the quantities $1+k$, ρ_2+k, $\cdots$, p_q+k is permitted to be a negative integer or zero.

The following dominating expression,

$$|\,\overline{\mathfrak{F}}_{k+m}(x)\,| \leqq |\,x\,|^{k+m}M,$$

where M is a positive constant, valid for all values of m and for all values of x in any given finite region, is readily obtained by employing the arguments of the previous article when $p=q$, and a fortiori when $p<q$. By means of this dominating expression, the expansion

$$(38)\quad x^{k+m}=\sum_{s=0}^{\infty}(-1)^s\,\frac{(a_1+k+m)_s\cdot(a_2+k+m)_s\cdots(a_p+k+m)_s}{(1+k+m)_s\cdot(\rho_2+k+m)_s\cdots(\rho_q+k+m)_s\quad s!}\,\overline{\mathfrak{F}}_{k+m+s}(x)$$

obtained as before, is shown to be absolutely convergent for all values of x in any finite region; and after dividing out the factor x^k, it is shown to be uniformly convergent for all values of x in any given finite region.

We again have

$$\frac{x^k}{t-x}=\sum_{m=0}^{\infty}\frac{x^{k+m}}{t^{m+1}}$$

$$=\sum_{m=0}^{\infty}\frac{1}{t^{m+1}}\sum_{s=0}^{\infty}(-1)^s\,\frac{(a_1+k+m)_s\cdot(a_2+k+m)_s\cdots(a_p+k+m)_s}{(1+k+m)_s\cdot(\rho_2+k+m)_s\cdots(\rho_q+k+m)_s\quad s!}\,\overline{\mathfrak{F}}_{k+m+s}(x)$$

$$(39)\quad =\sum_{m=0}^{\infty}\mathfrak{G}_{m,k}(t)\cdot\overline{\mathfrak{F}}_{k+m}(x),\qquad (p\leqq q),$$

where we define the polynomial

$$(40)\quad \mathfrak{G}_{m,k}(t)$$

$$=\sum_{s=0}^{m}\frac{(-1)^s}{s!}\cdot\frac{[a_1+k+m-1]_s\cdot[a_2+k+m-1]_s\cdots[a_p+k+m-1]_s}{[k+m]_s\cdot[\rho_2+k+m-1]_s\cdots[\rho_q+k+m-1]_s}\cdot\frac{1}{t^{m-s+1}}.$$

The expansion (39) is valid and converges absolutely for all values of x in any given finite region, assuming $|\,x\,|<|\,t\,|$, and converges uniformly when

$$|\,x\,|\leqq R',\quad |\,t\,|\geqq R,\quad \text{where } R'<R.$$

Writing $\overline{\mathfrak{F}}_{k+m}(x)=x^m\,\mathfrak{F}_{k+m}(x)$, we can state the theorems corresponding to Theorems IV, V, and VI, keeping in each the restriction that none of the quantities $1+k$, ρ_2+k, ρ_3+k, $\cdots$, ρ_q+k may be a negative integer or zero:

THEOREM VII. *Let x and t be two complex variables such that*

$$|x| \leqq R' \text{ and } |t| \geqq R,$$

where R is any finite number and $R' < R$; then the expansion

$$\frac{1}{t-x} = \sum_{m=0}^{\infty} \mathfrak{G}_{m,k}(t)\, \tilde{\mathfrak{F}}_{k+m}(x) \tag{41}$$

is valid. If t be a variable point on the contour C formed by the circle $|t| = R$, and x any fixed point in the circle $|x| = R'$, then the expansion converges uniformly with regard to t.

THEOREM VIII. *Let $f(x)$ be a function of x which is analytic and one-valued inside and upon the contour C formed by the circle $|x| = R$, where R is any finite number, then the expansion*

$$x^k f(x) = \sum_{m=0}^{\infty} a_m\, \bar{\mathfrak{F}}_{k+m}(x), \quad \text{where } a_m = 1/2\pi i \int_C \mathfrak{G}_{m,k}(t) f(t)\, dt, \tag{42}$$

*is valid when $|x| \leqq R' < R$; and the expansion divided by the factor x^k converges uniformly with regard to x when $|x| \leqq R'$.**

THEOREM IX. *If $f(x)$ is an analytic and one-valued function of x in the ring-shaped region defined by the inequalities*

$$0 < r \leqq |x| \leqq R,$$

where R is any finite number, the expansion

$$f(x) = \sum_{m=0}^{\infty} a_m\, \tilde{\mathfrak{F}}_{k+m}(x) + \sum_{m=0}^{\infty} a'_m\, \mathfrak{G}_{m,k}(x), \tag{43}$$

where $a_m = 1/2\pi i \int_C \mathfrak{G}_{m,k}(t) f(t)\, dt$, and $a'_m = 1/2\pi i \int_c \tilde{\mathfrak{F}}_{k+m}(t) f(t)\, dt$,

is valid and converges uniformly with regard to x when

$$r' \leqq |x| \leqq R',$$

where $r < r'$ and $R' < R$, and where C and c are the contours formed by the circles $|x| = R'$ and $|x| = r'$, respectively.

The sets of formulae corresponding, respectively, to (34) and (35) are

* This is a slight extension of a theorem derived in a different way by N. Nielsen, *Théorie des fonctions métasphériques* (Paris, 1911), page 165.

$$(44)\quad \begin{cases} a_m = \sum_{n=0}^{\infty} \dfrac{[a_1+k+m-1]_n \cdot [a_2+k+m-1]_n \cdots [a_p+k+m-1]_n}{[k+m]_n \cdot [\rho_2+k+m-1]_n \cdots [\rho_q+k+m-1]_n \quad n!} b_{m-n}, \\ a_0 = b_0 \qquad\qquad (m = 0, 1, 2, \cdots), \end{cases}$$

and

$$(45)\quad \begin{cases} a'_m = \sum_{n=0}^{\infty} \dfrac{(a_1+k+m)_n \cdot (a_2+k+m)_n \cdots (a_p+k+m)_n}{(1+k+m)_n \cdot (\rho_2+k+m)_n \cdots (\rho_q+k+m)_n \cdot n!} b'_{m+n+1}, \\ \qquad\qquad\qquad\qquad (m = 0, 1, 2, \cdots), \\ a_m = \sum_{n=0}^{m} \dfrac{[a_1+k+m-1]_n \cdot [a_2+k+m-1]_n \cdots [a_p+k+m-1]_n}{[k+m]_n \cdot [\rho_2+k+m-1]_n \cdots [\rho_q+k+m-1]_n \cdot n!} b_{m-n}. \end{cases}$$

Let L_1 be any finite closed curve not passing through the origin, then we have the following orthogonality properties:

$$(46)\quad \begin{aligned} &\int_{L_1} \tilde{\mathfrak{F}}_{k+m}(x) \cdot \tilde{\mathfrak{F}}_{k+n}(x)\, dx = 0, && (m = n \text{ and } m \neq n), \\ &\int_{L_1} \mathfrak{G}_{m,k}(x) \cdot \mathfrak{G}_{n,k}(x)\, dx = 0, && (m = n \text{ and } m \neq n), \\ &\int_{L_1} \tilde{\mathfrak{F}}_{k+n}(x) \cdot \mathfrak{G}_{n,k}(x)\, dx = 2\pi i\kappa, && (L_1 \text{ enclosing origin}), \\ &\qquad\qquad\qquad = 0, && (L_1 \text{ not enclosing origin}), \\ &\int_{L_1} \tilde{\mathfrak{F}}_{k+m}(x) \cdot \mathfrak{G}_{n,k}(x)\, dx = 0, && (m \neq n). \end{aligned}$$

It is well to remember that a desirable specialization of all of these expansions is obtained by making k equal to zero. This last general expansion where $p < q+1$ gives in the special case, when we take $k = 0$, $p = 0$, and $q = 1$, an expansion in terms of Bessel functions. For from the obvious relation

$$\tilde{\mathfrak{F}}_{0+m}\left(-\frac{x^2}{4}\right) = \left(-\frac{x^2}{4}\right)^m {}_0F_1\left(m+1; -\frac{x^2}{4}\right) = \frac{m!}{(-2)^m} x^m J_m(x),$$

where $J_m(x)$ is the Bessel function of order m, we have, by putting $x = \sqrt{z}$,

$$\tilde{\mathfrak{F}}_{0+m}\left(-\frac{z}{4}\right) = \frac{m!}{(-2)^m} (\sqrt{z})^m J_m(\sqrt{z}).$$

Hence, for an arbitrary function of z analytic and one-valued in any finite region inclosing the origin, we have the expansion in terms of Bessel functions,

$$f(z) = \sum_{m=0}^{\infty} \beta_m (\sqrt{z})^m J_m(\sqrt{z}),$$

which was first discussed by Nielsen.*

* N. Nielsen, *Handbuch der Theorie der Cylinderfunktionen* (Leipzig, 1904), p. 266.

IV. EXPANSIONS OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES.

20. The above developed expansion theory can easily be extended to the expansion of arbitrary analytic functions of several variables in a multiple series of hypergeometric functions. For the sake of simplicity, the case of two variables only will be developed, and the result then stated for functions of λ variables.

Assume that k is not equal to a negative integer. Let x_1, t_1 and x_2, t_2 be two pairs of complex variables in the x_1 and x_2-planes, respectively, such that

$$\text{(A)} \qquad \begin{aligned} |x_1| &\leqq r_1, \quad |t_1| \geqq R_1, \quad r_1 < R_1 \leqq 1, \\ |x_2| &\leqq r_2, \quad |t_2| \geqq R_2, \quad r_2 < R_2 \leqq 1; \end{aligned}$$

then, by Theorem I, the expansions

$$\text{(B)} \qquad \frac{1}{t_1 - x_1} = \sum_{m_1=0}^{\infty} G_{m_1,k}(t_1) \cdot \tilde{F}_{k+m_1}(x_1),$$

$$\text{(C)} \qquad \frac{1}{t_2 - x_2} = \sum_{m_2=0}^{\infty} G_{m_2,k}(t_2) \cdot \tilde{F}_{k+m_2}(x_2)$$

are valid. If t_1 and t_2 are variable points, respectively, on the contours C_1 and C_2 formed by the circles $|t_1| = R_1$ and $|t_2| = R_2$, and if x_1 and x_2 are any fixed points in the circles $|x_1| = r_1$ and $|x_2| = r_2$, respectively, then the expansions (B) and (C) converge uniformly with regard to t_1 and t_2 respectively.

Since both expansions are absolutely and uniformly convergent when the conditions in (A) are fulfilled, then their product,

$$(47) \qquad \frac{1}{t_1 - x_1} \cdot \frac{1}{t_2 - x_2} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} G_{m_1,k}(t_1) \cdot G_{m_2,k}(t_2) \cdot \tilde{F}_{k+m_1}(x_1) \cdot \tilde{F}_{k+m_2}(x_2),$$

is absolutely and uniformly convergent under the same conditions.

Now let $f(x_1, x_2)$ be an arbitrary function of the two independent complex variables x_1 and x_2, which is analytic and one-valued when x_1 and x_2 are, respectively, in the regions bounded by the contours C_1 and C_2; then, by Cauchy's generalized integral formula, we have

$$f(x_1, x_2) = (1/2\pi i)^2 \int_{C_1} \int_{C_2} \frac{f(t_1, t_2)\, dt_2 dt_1}{(t_1 - x_1)(t_2 - x_2)}$$

$$= (1/2\pi i)^2 \int_{C_1} \int_{C_2} \left[\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} G_{m_1,k}(t_1) \cdot G_{m_2,k}(t_2) \cdot \tilde{F}_{k+m_1}(x_1) \cdot \tilde{F}_{k+m_2}(x_2) \right]$$

$$\times \frac{f(t_1, t_2)\, dt_2 dt_1}{(t_1 - x_1)(t_2 - x_2)}$$

$$(48) \qquad = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A_{m_1,m_2} \tilde{F}_{k+m_1}(x_1) \cdot \tilde{F}_{k+m_2}(x_2),$$

where $$A_{m_1,m_2} = (1/2\pi i)^2 \int_{C_1} \int_{C_2} f(t_1, t_2)\, G_{m_1,k}(t_1) \cdot G_{m_2,k}(t_2)\, dt_2 dt_1.$$

This expansion is valid and converges uniformly with regard to x_1 and x_2 when x_1 and x_2 are, respectively, in the circles $| x_1 | = r_1$ and $| x_2 | = r_2$.

Since the extension to any number of variables is evident, we shall state the result for λ variables:

THEOREM X. *Let $f(x_1, x_2, \cdots, x_\lambda)$ be a function of $x_1, x_2, \cdots, x_\lambda$, which is analytic and one-valued when $x_1, x_2, \cdots, x_\lambda$ are, respectively, inside and upon the contours $C_1, C_2, \cdots, C_\lambda$ formed by the circles $| x_1 | = R_1 \leqq 1$, $| x_2 | = R_2 \leqq 1, \cdots, | x_\lambda | = R_\lambda \leqq 1$, then the expansion*

$$(49) \quad f(x_1, x_2, \cdots, x_\lambda) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_\lambda=0}^{\infty} A_{m_1,m_2 \ldots, m_\lambda} \tilde{F}_{k+m_1}(x_1) \cdot \tilde{F}_{k+m_2}(x_2) \cdots \tilde{F}_{k+m_\lambda}$$

where

$$A_{m_1,m_2 \ldots, m_\lambda} = (1/2\pi i)^\lambda \int_{C_1} \int_{C_2} \cdots$$

$$\cdot \int_{C_\lambda} f(t_1, t_2, \cdots, t_\lambda) \cdot G_{m_1,k}(t_1) \cdot G_{m_2,k}(t_2) \cdots G_{m_\lambda,k}(t_\lambda) \cdot dt_\lambda \cdots dt_2 dt_1,$$

is valid and converges uniformly, when $x_1, x_2, \cdots, x_\lambda$ are, respectively, in the circles $| x_1 | = r_1$, $| x_2 | = r_2, \cdots, | x_\lambda | = r_\lambda$, where

$$r_1 < R_1, \quad r_2 < R_2, \cdots, r_\lambda < R_\lambda,$$

provided k is not equal to a negative integer.

It is evident that corresponding theorems for the expansion of functions of several variables in multiple series of each of the two classes of generalized hypergeometric functions can also be stated, merely replacing the functions $\tilde{F}_{k+m_i}(x_i)$, $G_{m_i,k}(t_i)$ in turn by the functions $\tilde{\mathfrak{F}}_{k+m_i}(x_i)$, $\mathfrak{G}_{m_i,k}(t_i)$ and $\tilde{\mathfrak{F}}_{k+m_i}(x_i)$, $\mathfrak{G}_{m_i,k}(t_i)$, and noting that for the second class the circles of convergence may have any finite radii.

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